PATTERN EQUATION METHOD FOR SOLUTION OF ELECTROMAGNETIC SCATTERING PROBLEMS*

A.G.Kyurkchan, S.A.Manenkov

Moscow Technical University of Communication and Information Science

Aviamotornaya 8a, Moscow, 111024, Russia

E-mail: kurkchan@cityline.ru

A.I.Kleev

P.L.Kapitza Institute for Physical Problems

Kosygina 2, Moscow, 117334, Russia

Fax: 7(095) 9382030

E-mail: kleev@kapitza.ras.ru

A.I.Sukov

Mathematical Department

Moscow State Technological University STANKIN

Vadkowski per., 3a, Moscow, 101472, Russia

Abstract

The Pattern Equation Method (PEM) is a new numerical-analytical method of solving of diffraction problems. The method is based on the modifications of the plane wave representations of the diffraction field by the Sommerfeld-Weil integrals which have been previously obtained by A.G.Kyurkchan. These modifications of the classic plane wave integrals permit to find the diffraction field by its asymptotic in far zone that is by the pattern. The field can be found in the whole space outside the field singularities convex envelope which is located entirely inside the scatterer in most interesting cases for practice.

1. PROPOSED METHOD

The Pattern Equation Method (PEM) has been presented in [1] and then tested in papers [2-15,17]. It is the new rigorous and universal method of solving boundary problems of electrodynamics, acoustics and optics. This method reduces to neither standard methods of solving mathematical physics boundary problems. In solving problems of wave scattering by single body [1-3] the group of bodies [4, 5], discrete

gratings in free space [6], periodical interface of two media [7, 8], a body immersed in homogeneous semi-space [9-11] and a body in layered medium as well as the problems of modes of periodical structures [13, 14] and dielectric waveguides [15] the high efficiency has been demonstrated.

We shall state the main idea of the method using two dimensional problem of the primary plane wave field U^0 diffraction on the cylindrical scatterer with the cross-section S [1, 7]. When a two-dimensional problem is considered the representation mentioned above has the following form [16]:

$$U^1(r,\varphi) = \frac{1}{\pi} \int_{-\frac{\pi}{2}-i\infty}^{\frac{\pi}{2}+i\infty} g(\varphi+\psi)\exp(-ikr\cos\psi)d\psi \tag{1}$$

where U^1 is the wave (diffraction) field which satisfies the homogeneous Helmholtz equation everywhere outside the scatterer:

$$\Delta U^1 + k^2 U^1 = 0,$$

$g(\varphi)$ is the pattern of the wave field, that is the function of the angle coordinate in asymptotic equality:

$$U^1(r,\varphi) = \sqrt{\frac{2}{\pi kr}}\exp\left(-ikr + i\frac{\pi}{4}\right)g(\varphi) + O\left(r^{-3/2}\right).$$

The pattern $g(\varphi)$ in turn is expressed in terms of the field $U^1(r,\varphi)$ by the following integral:

$$g(\alpha) = \frac{i}{4}\int_S \left\{\frac{\partial U}{\partial n}\exp\left[ikr\cos(\alpha-\varphi)\right]\right\}\Bigg|_S dS \tag{2}$$

where $U = U^0 + U^1$, U^0 is the primary field. For simplicity we restrict ourselves to the case when the complete wave field satisfies the homogeneous Dirichlet condition on the scatterer surface S. Based on the above mentioned relations (1) and (2) present the pair of transformations which allows to semiinvert the diffraction operator and reduce the boundary problem to a second kind integrooperator equation relative to the pattern. The example of the equation is [1]:

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$$g(\alpha) = g^0(\alpha) + \frac{1}{\pi}\int_{-\frac{\pi}{2}-i\infty}^{\frac{\pi}{2}+i\infty} \hat{K}(\alpha,\psi) g(\psi) d\psi. \tag{3}$$

Here $g^0(\alpha)$ is the known function, $\hat{K}(\alpha,\psi)$ is the operator function,

$$\hat{K}(\alpha,\psi) = \frac{k}{4}\int_0^{2\pi} \left[\rho(\varphi)\cos\psi - \rho'(\varphi)\sin\psi\right] \exp\left\{-ik\rho(\varphi)\left[\cos\psi - \cos(\alpha-\varphi)\right] + \varphi\frac{d}{d\psi}\right\} d\varphi$$

where $r = \rho(\varphi)$ is the equation of the contour S in the polar coordinates.

In the case of a three-dimensional problem the following modification of the plane wave integral is used as the main relation [2]:

$$U^1(r,\theta,\varphi) = \frac{\exp(-i\pi/4)}{\sqrt{2\pi kr}} \int_{-\frac{\pi}{2}-i\infty}^{\frac{\pi}{2}+i\infty} \exp(-ikr\cos\psi) \exp\left[-i\psi\left(\hat{B} + \tfrac{1}{2}\right)\right] g(\theta,\varphi) d\psi, \tag{4}$$

where $g(\theta,\varphi)$ - is the pattern of the wave field; $\hat{B}$ is the operator determined by the formula:

$$-\hat{B}(\hat{B}+1) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

If the pattern is defined by its Fourier series:

$$g(\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{nm} P_n^m(\cos\theta)\exp(im\varphi), \tag{5}$$

then:

$$\exp(i\psi\hat{B}) g(\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{nm} P_n^m(\cos\theta)\exp(im\varphi + in\psi).$$

Using the formula (4) as well as the integral representation for the pattern $g(\theta,\varphi)$ in terms of the boundary value of the wave field and its normal derivative we can get integro-operator equation of the second kind relative to the function $g(\theta,\varphi)$ similar to (3) [2]. The integral equations in both two- and three-dimensional cases are solvable providing the convex envelope of the diffraction field singularities lies entirely inside the scatterer. All convex as well as so called slightly nonconvex bodies satisfy this condition. Expanding the pattern in terms of some complete system of functions, for

example, into the Fourier series, we can reduce the equation (3) to the infinite algebraic system:

$$a_n = a_n^0 + \sum_{m=-\infty}^{\infty} G_{nm} a_m \tag{6}$$

relative to the Fourier coefficients a_n of the pattern. The free terms a_n^0 correspond to the expansion of the function $g^0(\varphi)$, G_{nm} are the elements of the system matrix.

The Pattern Equation Method has a number of advantages compared with the standard methods. These advantages are conditioned by the nature of the method. Indeed, when the boundary problem is solved by the current integral equation method, the required value is the distribution of the tangential component of the wave field (that is the current) on the scatterer surface. This value strongly depends on the body geometry. For example, the currents at the surface of a sphere and a thin disk with the same diameter have very different dependencies versus the coordinates. At the same time the main parameters characterizing the pattern such as the main-lobe beam width, the first side beam level etc. slightly depend on the geometry and are mostly determined by the typical sizes of the scatterer. Another example is the problem of wave scattering by two bodies. If dimensions of the bodies are small compared with the wavelength, their patterns are approximately isotropic for all distances between the scatterers (in a two-dimensional case). However the currents on the bodies surfaces significantly change when scatterers are close to each other. Thus the dimensions of the algebraic systems occurring when the problem is solved by Pattern Equation Method are determined by the typical sizes of the scatterer and practically don't depend on its geometry, the type of the boundary conditions and the availability of other bodies near by.

TABLE I

N	7	8	9
$cond$	~ 1	~ 1	~ 1
$S_i/2\pi a^2$	1,033	1,033	1,033
$\lvert 1 - S_0/S_i \rvert$	$8 \cdot 10^{-3}$	$6 \cdot 10^{-4}$	$4 \cdot 10^{-4}$

The table I illustrates the calculation results of the scattering characteristics for a oblate spheroid with the semiaxis $ka = kb = 5$, $kc = 0.5$. Here $cond$ is the system matrix condition number; S_i is the integral scattering cross section; S_0 is the value of the integral scattering cross-section obtained from the optical theorem; N is the maximum number of the harmonics in the series of type (5).

You can see that the results are stabilized in third decimal place when $N \approx kd$ where d is the diameter of the sphere circumscribed around the spheroid. So the PEM "doesn't observe" such a significant sphere deformation.

The integral equation for the pattern permits to obtain explicit analytical solutions for a number of interesting practical cases. One of these examples is presented below. This is the second merit of the method.

The third advantage of the PEM is that the matrix elements of the algebraic systems, generated by the method, are expressed by the integrals half of dimension as much as those in the standard methods, for example, in the method of current integral equations.

To illustrate the next advantage of the method we present the following form of the matrix of the algebraic system obtained in solving the problem of wave diffraction by periodical grating which is located near to the two media interface:

$$\mathbf{G} = \mathbf{G}^{\mathbf{s}} + \mathbf{G}^{\mathbf{a}}(\mathbf{G}^{\mathbf{g}} + \mathbf{G}^{\mathbf{h}}). \tag{7}$$

You can see from (7) that the matrix of the algebraic system is the sum of two matrices. The first of these matrices is similar to that used in solving the problem of wave scattering by a single scatterer which is the element of the grating in the problem considered. The second matrix is the multiplication of the two matrices: $\mathbf{G}^{\mathbf{a}}$ and $(\mathbf{G}^{\mathbf{g}} + \mathbf{G}^{\mathbf{h}})$. Here $\mathbf{G}^{\mathbf{a}}$ is determined just by the geometry of the grating element and the type of the boundary condition on its surface as well as the matrix $\mathbf{G}^{\mathbf{s}}$. The matrix $\mathbf{G}^{\mathbf{g}}$ depends only on the parameters characterizing the grating itself (the period and the incidence angle of primary plane wave). The matrix $\mathbf{G}^{\mathbf{h}}$ is determined by the parameters of the grating, the parameters of the media and the distance between the grating and the interface.

Thus the matrices $\mathbf{G}^{s}$ and $\mathbf{G}^{a}$ are universal in the various problems of effort required and don't depend on the availability of any surfaces or medium inhomogeneities surrounding the scatterer.

At last one more important advantages of the PEM is the possibility of rigorous justification of the computational algorithm of solving one or another diffraction problem. There is a unified procedure to fulfill the asymptotical evaluations of the matrix elements and free terms of the corresponding algebraic systems. This procedure allows to get following sufficient condition to solve the systems by the reduction method:

$$\sigma_2 > \sigma_1 \tag{8}$$

The values σ_1 and σ_2 are determined by the problem geometry and the type of the primary field. The condition (8) is obviously fulfilled providing the scatterer is convex or slightly nonconvex and the interactive bodies don't touch each other.

The condition (8) rather strongly restricts the ratio of the "amplitude" of period of the periodical surface if the problem of scattering by the periodical surface is considered [7]. For example, in the case of wave diffraction by the sinusoidal surface $f(x) = a\cos(2\pi x/b)$, where a is the amplitude of the surface and b is it's period, the condition (8) leads to the limitation:

$$a/b < 0.1055\ldots$$

However the limitations on the problem geometry which follow from the inequality (8) don't make the problem insolvable. The problem of wave scattering by a body of complete geometry can be replaced by the problem of diffraction on the group of bodies near by with more simple shape.

Analogous replacement can be made in the case of the scattering by strongly "modulated" surface.

As an illustration of PEM applying to various problems we shall consider the concrete examples.

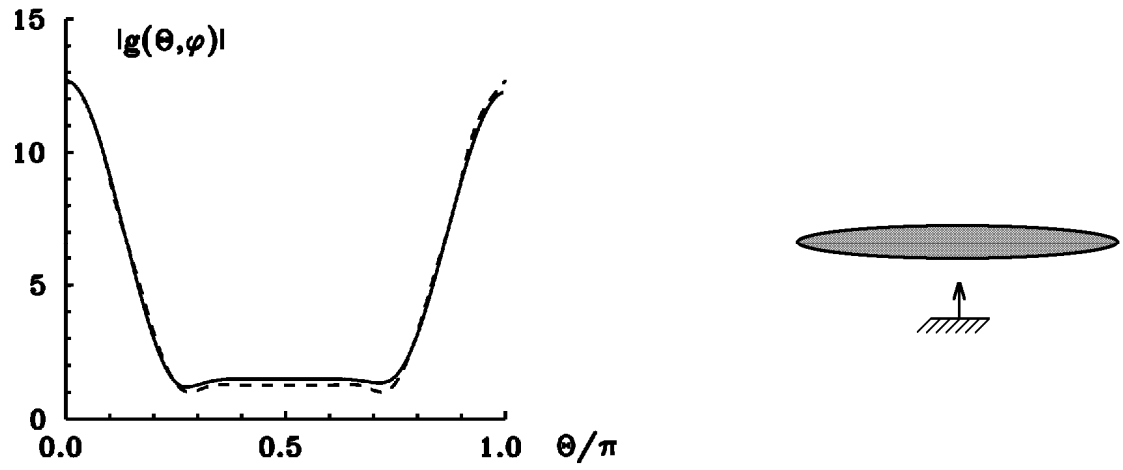


Figure 1.

The scattering pattern of a very oblate spheroid with the semi-axes $ka = kb = 5$, $kc = 0.125$ which has been found by PEM is shown on fig. 1. The normal plane wave incidence is assumed. The pattern of infinite thin disk with the radius $ka = 5$ is shown by a dashed line on this figure. You can see good coincidence of the curves. In calculations we used the equation system similar to (6) which had dimensions 11×11. So the system order is approximately equal to kd, where d is the largest cross size of the body. Further increase of the system dimension results in the changes of third or fourth decimal place.

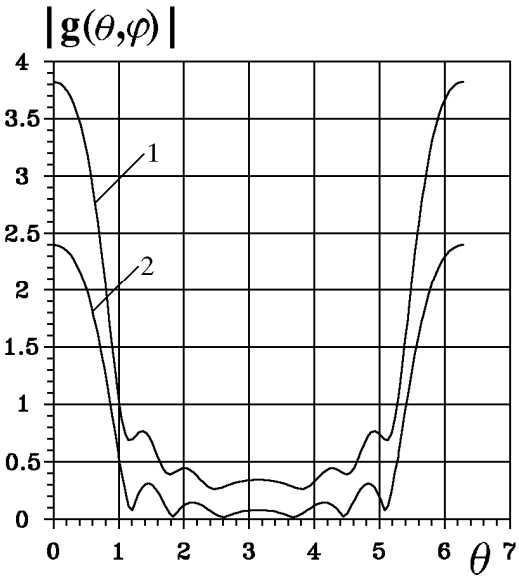


Figure 2.

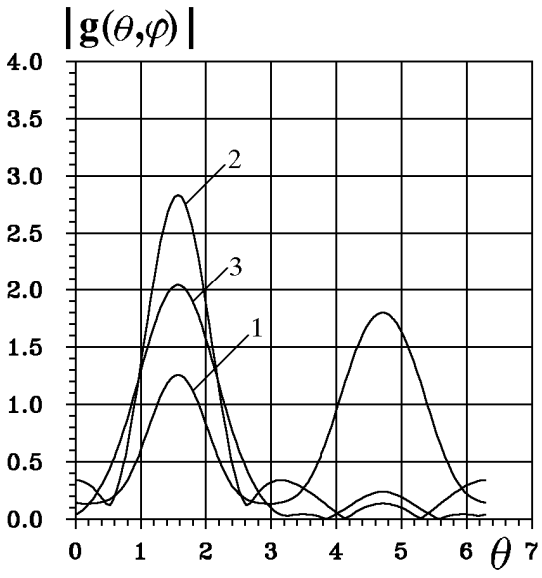


Figure 3.

Let's consider the problem of wave scattering by an impedance body [17]. Assume the so called impedance boundary condition is satisfied on the body surface:

$$U\big|_S = \frac{W}{k}\frac{\partial U}{\partial n}\bigg|_S$$

where W is proportional to the local impedance of the scatterer surface. Figure 2 presents the pattern of an prolate spheroid with the semiaxes $ka = kb = 1$, $kc = 5$ irradiated by the plane wave which propagates along the large semiaxis, that is $\theta_0 = 0$, $\varphi_0 = 0$. The curve 1 corresponds to the acoustically soft spheroid ($W = 0$), the curve 2 corresponds to the spheroid with the so called «matching impedance» ($W = -i$), that is the completely sound-absorbing spheroid. You can see that the reverse reflection (in the direction of $\theta = \pi$) by the ideal spheroid is more than 10dB lower than it is by the acoustically soft spheroid. The figure 3 shows the results of the calculation of the pattern of the plane wave propagating orthogonally the symmetrical axis of the body representing the round cylinder with the radius $ka = 1$ and the height $2kc = 6$ (the curves 1,2), or the spheroid with semiaxis $ka = kb = 1$, $kc = 3$ inscribed into the cylinder (the curve 3). In this case the curve 1 corresponds to the acoustically hard body ($W = \infty$) and the curves 2 and 3 correspond to the sound-absorbing body. In the situation considered the reverse reflected field attenuation for the completely absorbing scatterer is still more. Besides the comparison of the curves 2 and 3 demonstrates that the pattern typical minor lobes which are correspond to the end-walls of the cylinder ($\theta = 0,\ \pi$) are absent.

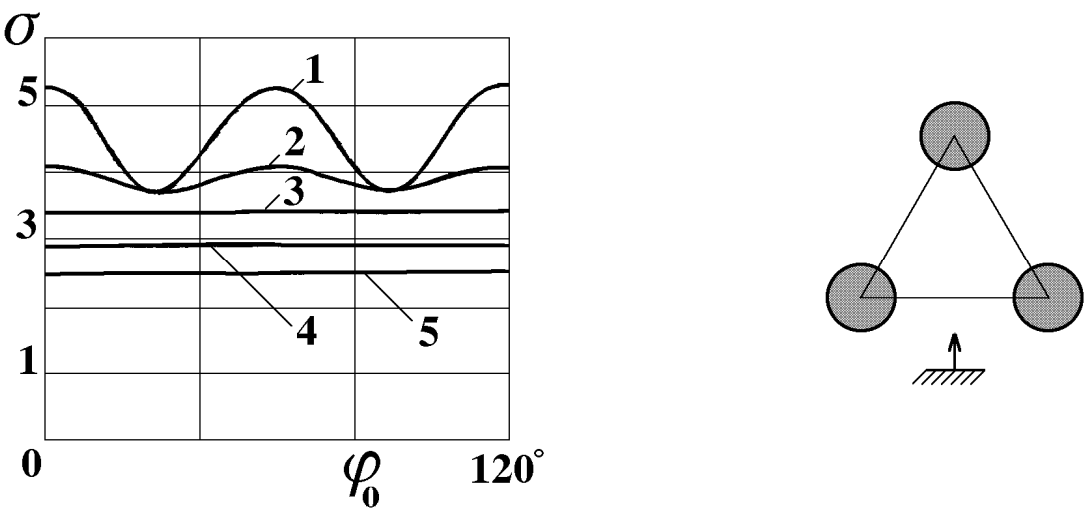


Figure 4.

As the second example we consider the problem of wave scattering by the group of N bodies. The dependence of the effective scattering cross-section (ESC) σ on the

plane wave incidence angle for the group of three round cylinders with the radius $ka = 1$ is shown on fig. 2. The cylinders are located at the apexes of equilateral triangle. The length of the triangle side varied between $kL = 8$ (the curve 1), 5 (the curve 2) ... 3 (the curve 4) and $kL = 2.04$ (the curve 5) [5]. You can see that with decreasing of the distance between the scatterers the integral ESC of the group σ_Σ monotonically diminishes and with increasing kL the ratio σ_Σ / N tends to the limit equal to σ_Σ of the single scatterer. Such a behavior of the value σ_Σ can be interpreted by the fact that in a "dense" group of scatterers some part of incident wave power remains in the system on the average per period.

And now consider an example of the PEM application to obtain some analytical relations. The following approximate formula for the reflection coefficient of the plane wave by the periodical grating composed of round cylinders was presented in the paper [6]:

$$R_0 \approx -\left[1 - i\left(\frac{2b}{\lambda}\ln\frac{2\pi a}{b}\right)\sin\varphi_0\right]^{-1}, \tag{9}$$

where a is the cylinder radius, b is the grating period, λ is the incident wavelength, φ_0 is the grazing angle. In spite of its simplicity this formula is valid for approximate calculations when $b/\lambda << 1$. The comparison with the rigorous computations using the system of type (6) shows that when $b/\lambda = 0.05$ the "exact" reflection coefficient doesn't differ from the approximate one for more than 0.1%.

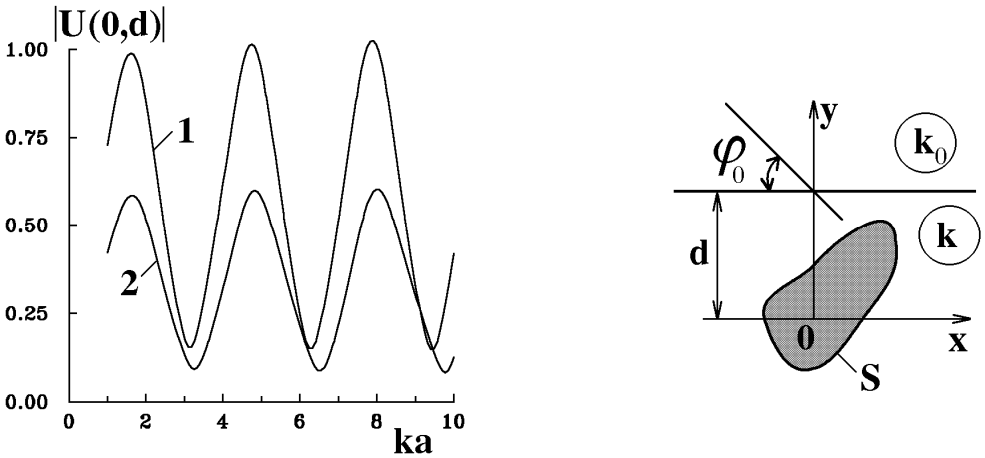


Figure 5.

The fig. 5 demonstrates the dependence of the absolute value of the field $U(0,d)$ versus the wave number for the cylindrical scatterer immersed in the semi-space to the depth $d = 2a$. The refraction index ν is equal to 3. It follows from the graph that the maximums of the dependence corresponding normal wave incidence (the curve 1) are at the points $ka = \pi/2 + \pi n, \quad n = 1,2,3\ldots$. Thus the irradiation of the media interface by a plane wave with various frequencies permits to evaluate a body immersion depth since the full field maximum corresponds to the value $d - a$ which is equal to $\lambda/4$. The curve 2 corresponds to the incident wave grazing angle $\varphi_0 = 30^\circ$.

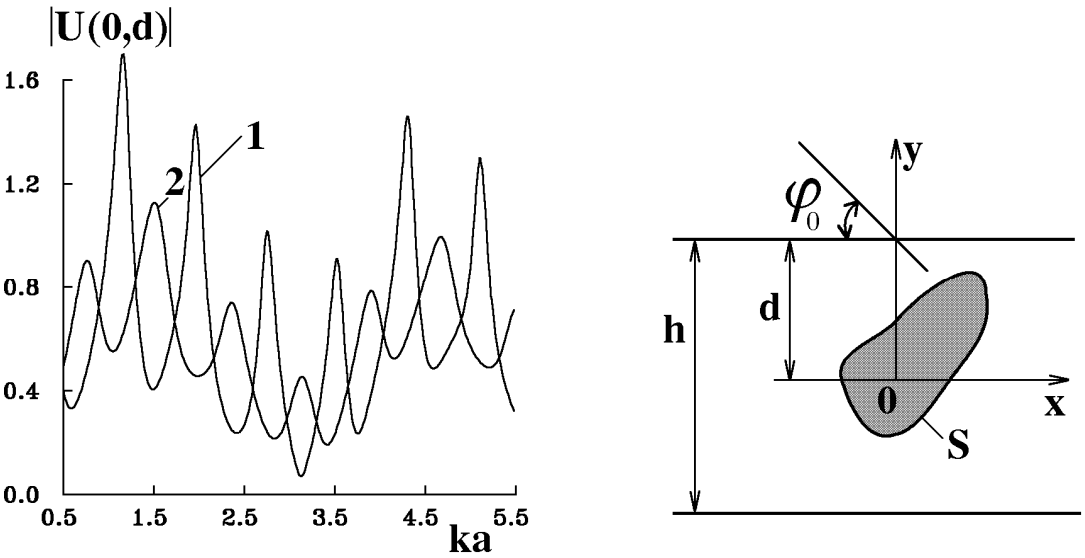


Figure 6.

On the next figure the distributions of the field $U(0,d)$ versus ka in the case when a scatterer is inside the plane layer [11] (see fig. 6) is shown. The thickness of layer is $h = 4a$. The curve 1 corresponds to the layer lying on the metallic screen and the curve 2 corresponds to the symmetrical layer, with $d = h/2$.. Maximums of the curves conform here to the critical frequencies of the modes propagating in the layer. For instance for the curve 1 they correspond to the values

$$k_n a = \frac{\sqrt{\nu^2 - 1}}{\nu} \frac{\pi(2n-1)}{8}, \; n = 1,2,\ldots.$$

The calculations show that the maximums of the dependences are not displaced if the distance d between the body and the upper layer boundary is changed. The refraction index ν is equal to 3.

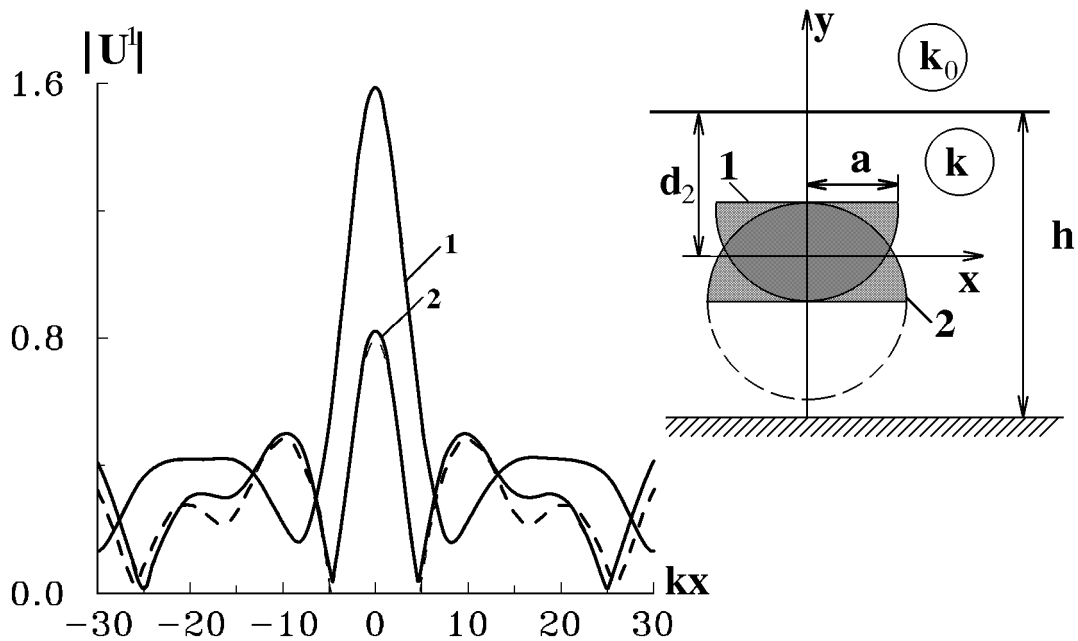


Figure 7.

Figure 7 presents the distribution of the scattering wave field on the upper boundary ($y = d_2$) of the homogeneous dielectric layer which lies on the metallic screen $y = -d_1$ ($d_1 = h - d_2$). At the top the layer borderes on the homogeneous semispace $y > d_2$. The scatterer is assumed to be a semicircle located within the layer and the primary field is a plane wave falling from the semispace $y > d_2$ (the normal incidence is supposed to be). The curves 1 and 2 in the figure correspond to two various positions of the scatterer (see the graph). It follows from the figure that the scattering field at the point $x = 0$ for the position 1 is approximately two times more than for the position 2. The dashed curve in the figure conforms to the scattering by the round cylinder with the same radius. The cylinder is placed so that the distance between the point on its surface closest to the interface $y = d_2$ and this layer boundary is equal to the corresponding distance for the semicircle (positions 1 and 2). The figure demonstrates good coincidence of the curve 2 and the dashed curve close to the point $x = 0$.

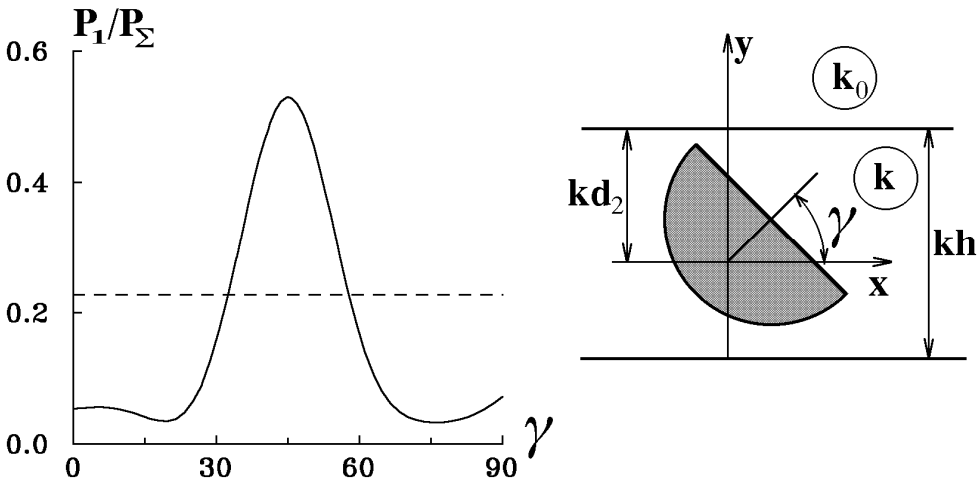
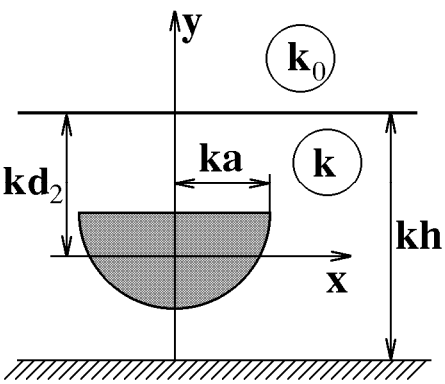


Figure 8.

To illustrate the effect of the excitation of the main (first) mode by the mirror with the semicircular cross-section we have plotted the dependence of the ratio between the value of the main mode power and the total mode power (carried along the positive direction) versus the rotation angle γ of the body (see fig. 8). The radius of the scatterer is $ka = 5$. The body is placed in the middle of the symmetric layer which is irradiated by the normal incident plane wave. The relative refraction coefficient is equal to 3.13.The maximum of the curve conforms to the point $\gamma = 45^\circ$. This result is in good agreement with the geometry optic theory. For comparison we have plotted the analogous dependence for the small round cylinder with the radius $ka = 0.5$ (the dashed straight line in the figure). It follows from the figure that the semicircular reflector is rather effective to excite the mane mode in this slab.

TABLE II



$\nu = 3.1305 \quad (\nu = k/k_0)$

$\frac{h}{a} = 4, \quad \varphi_0 = \frac{\pi}{2}$

ka	kd_2	N_u (2 dig.)	N_g (3 dig.)
0.5	0.56	7	2
1.5	1.68	12	4
3	3.36	25	6

The table II illustrates calculation algorithm convergence. The values N_u and N_g are the numbers of terms in Fourier series for field and pattern respectively. The quantities near these values indicates the calculation precision (number of significant digits).

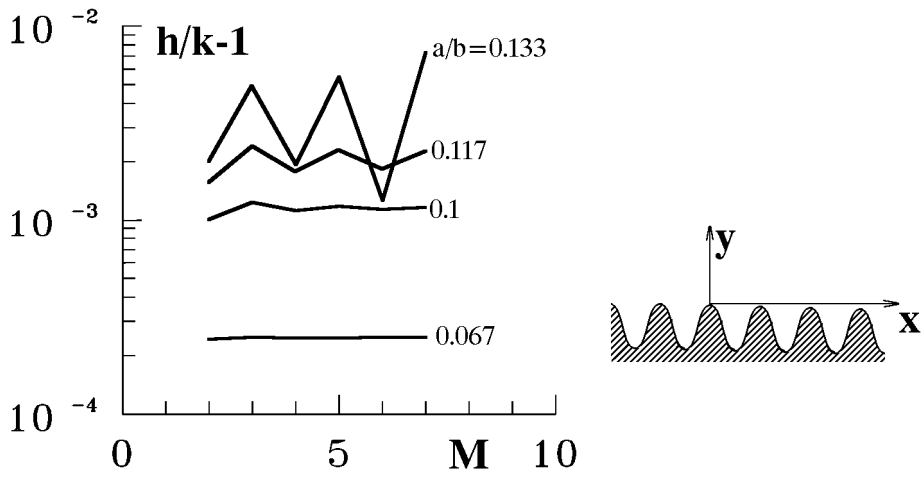


Figure 9.

Fig. 9 illustrates the result of PEM application for the problem of the eigen waves which are propagates along the sinusoidal surface $f(x) = a\cos(2\pi x/b)$ [7, 13, 14]. The dependence of the value $h/k - 1$ (where h is the longitude wave number) on the number of the equations in the system of the type (6) is shown on the figure. The dependencies are plotted for the various values of the parameter a characterizing the depth of the sinusoidal surface. The limitation (8) in this problem is equivalent to the condition $a/b < 0.1055\ldots$. The figures shows that when this condition is broken the algorithm becomes unstable.

In conclusion we present the results concerning the wave scattering by gratings. The pattern of the grating consisting of a round cylinders placed in free space is shown in the figure 10a. The radius of the grating elements a and the period of the grating b satisfy the relation: $a = 0.1b$ ($b = 0.75\lambda$, where λ is the wavelength). The grating is excited by the point source located at $x_0 = 0$, $y_0 = 2a + b$. The figure 10b illustrates the pattern of the same grating which is immersed in the homogeneous semispace with the relative refraction coefficient $\nu = 3$. The depression of all spectral maximums of the large numbers is caused by the fact that the grazing angles of the

high number waves are less than the total internal reflection angle of the interface $y = d$. Thus the interface becomes opaque for these waves.

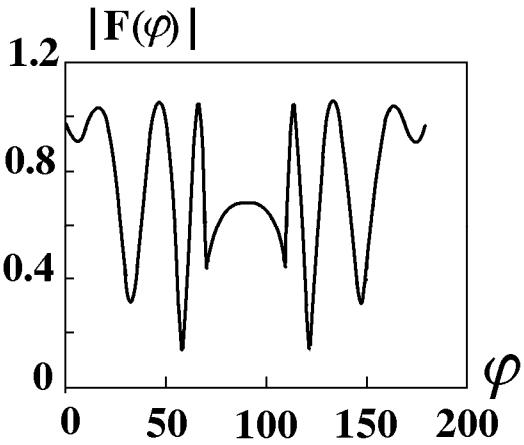


Figure 10a.

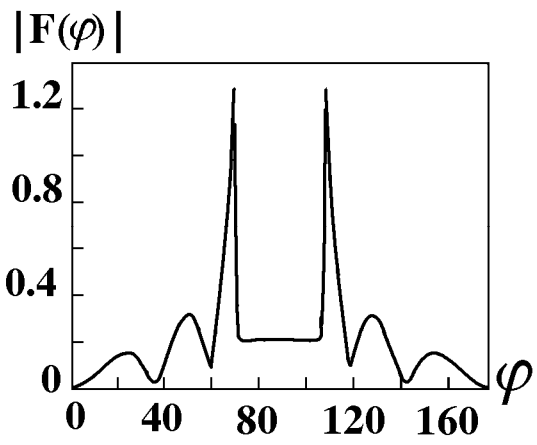


Figure10b

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