SOLUTIONS OF NONLINEAR INTEGRAL EQUATION OF SYNTHESIS OF THE LINEAR ANTENNA ARRAYS

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Abstract

The problem of synthesis of the equidistant linear antenna array according to the prescribed amplitude pattern, reduced formerly to a nonlinear integral equation, is solved analytically with a small number of unknown complex parameters. For these parameters, a set of transcendental equations is obtained. Limitation of the number of parameters is given. Numerical results are presented. The results obtained can be generalized for other problems described by the discrete Fourier transformation.

1. INTRODUCTION

The synthesis problems of different kind of antennas according to prescribed radiation pattern or to its different physical characteristics are widely discussed in the literature [1-6]. For the practically important case, if only amplitude radiation pattern is required, these problems have been described in [7]. The problems were stated in the variational formulation as problems of minimization of the mean-square difference between the given and obtained amplitude patterns in the corresponding functional spaces. These last problems were reduced to nonlinear integral or matrix equations which have been solved numerically by the iterative methods.

Recently, explicit solutions of such kind of equations have been obtained for the case of the linear antenna [8]. In this case the radiation pattern is calculated as the Fourier transform of the current distribution at the antenna. The solutions have the form of integral expressions depended on a finite numbers of complex parameters for which a finite system of transcendental equations are obtained. These results can also be used in other applications described by the Fourier transformation.

In this paper the results mentioned above are extended for the case of equidistant linear antenna arrays described by the discrete Fourier transformation. The main distinction of these problem in comparison with preceding case is connected with periodicity of the radiation pattern as a function of generalized coordinates, and following this, with the way of introducing the complex parameters in question. In part, the results described were presented in [9].

2. PROBLEM STATEMENT

As it is well known, the radiation pattern of the equidistant linear antenna array with identical radiators having the individual patterns $f_0(\vartheta)$ is calculated by the formula

$$f(\vartheta) = f_0(\vartheta) \sum_{n=-M}^{M} I_n e^{ikdn\sin\vartheta} \tag{1}$$

where ϑ is the polar angle in the spherical coordinate system, N=2M+1 is the number of elements in the array (for simplicity, N is assumed to be the odd number), I_n is the current at the nth elements, k is the wave factor, and d is the distance between the elements. The mutual influences of the elements is neglected in this formula.

Let us consider a real positive function $F(\vartheta)$ being the prescribed amplitude directivity pattern which should be approximated by the actual amplitude directivity pattern $|f(\xi)|$. It is natural to assume that $F(\vartheta) \equiv 0$ in the side lobes area that means for any $|\vartheta| > \alpha$ where 2α is the main lobe width. In what follows we introduce a generalized angle coordinate $\xi = \sin\vartheta / \sin\alpha$ and will denote the prescribed and actual radiation patterns in these coordinate by the same symbols F and f, respectively. For the case $f_0(\xi) \equiv 1$, in new coordinates formula (1) gets the form

$$f(\xi) = \sum_{n=-M}^{M} I_n e^{icn\xi} \tag{2}$$

where $c = kd\sin\alpha$ is a specific physical parameter of the problem. Accordingly, the prescribed amplitude pattern $F(\xi)$ equals 0 for $|\xi| > 1$.

One can see that in the form (2) the radiation pattern $f(\xi)$ is a periodical function of ξ with the period $2\pi / c$. It is sensible to prescribe the amplitude pattern $F(\xi)$ only on one period, what gives a inequality $c \le \pi$. Then the synthesis problem for the equidistant linear antenna array with identical elements can be formulated as the problem of minimizing the functional

$$\sigma = \int_{-\pi/c}^{\pi/c} \left(F(\xi) - |f(\xi)| \right)^2 d\xi \,. \tag{3}$$

The vector $\mathbf{I} = \{I_n\}$ of the current distribution at the antenna elements is the subject to be found in this problem. In [7] this problem was reduced to the nonlinear integral equation

$$f(\xi) = \int_{-1}^{1} K(\xi - \xi')F(\xi')e^{i\arg f(\xi')}d\xi' \tag{4}$$

where

$$K(\xi - \xi') = \frac{2\pi}{c}\frac{\sin(Nc(\xi - \xi')/2)}{(\sin c(\xi - \xi')/2)}. \tag{5}$$

When this equation is solved, the desired current distribution is calculated by the formula

$$I_n = \frac{c}{2\pi}\int_{-1}^{1} e^{-inc\xi}F(\xi)e^{i\arg f(\xi)}d\xi, \tag{6}$$

$$n=-M,...,M.$$

Equation (4) is the Euler equation for functional (3). Its solutions are not only minimum points of the functional, but also all its stationary ones.

Equation (4) is the subject to investigation in this paper. As it is established in [7], this equation has nonunique solutions. At some values of parameter c the branching of the solutions take place and number of these solutions increases. To separate the optimal solutions among them, they should be substituted into functional σ and values obtained should be compared.

Recall some simple properties of equation (4) from [7]:

i. If the function

$$f(\xi) = \int_{-1}^{1} K(\xi - \xi')F(\xi')d\xi' \tag{7}$$

is positive at $-1 < \xi < 1$, then it solves equation (4);

ii. If a function $f(\xi)$ solves (4), then the complex-conjugated function $f^*(\xi)$, as well as $f(\xi)\exp(i\alpha)$ with any real α solves (4), too.

In [7] equation (4) was solved numerically by an iterative method and branching of its solutions was investigated by the methods of nonlinear analysis [10]. In what follows solutions of (4) will be expressed in the analytical form containing a finite number of complex parameters which can be calculated from a system of transcendental equations.

3. ANALYTICAL SOLUTIONS OF THE PROBLEM

It is seen from (4) that any solution of this equation is uniquely defined by its phase. This means that it is sufficient for solving (4) to find the function $\exp(i\arg f(\xi))$. Suppose, that this function can be expressed in the form

$$e^{i\arg f(\xi)} = \frac{P_L(\tau)}{|P_L(\tau)|} \tag{8}$$

where $\tau = \tan c\xi/2$,

$$P_L = (1-\eta_1\tau)(1-\eta_2\tau)\ldots(1-\eta_L\tau) \tag{9}$$

η_n are some complex parameters non conjugated pairwise: $\eta_n - \eta_m^* \neq 0$, n,m = 1, 2, ..., L, and number L is some finite integer.

Substituting (8) into (4) and separating the integer parts from both hand sides of the equation yields

$$\int_{-1}^{1} K(\xi-\xi')F(\xi')\frac{\mathrm{Im}\left(P_L(\tau)P_L^*(\tau')\right)}{|P_L(\tau)|}d\xi' \equiv 0 \tag{10}$$

It is easily seen that

$$\sin\frac{c(\xi-\xi')}{2} = (\tau-\tau')\cos\frac{c\xi}{2}\cos\frac{c\xi'}{2} \tag{11}$$

Using this equality in $K(\xi-\xi')$ and substituting (9) into the numerator of (10), we have

$$\sum_{n,m=1}^{L} d_{nm}(\tau')^{m-1}\left[\frac{\cos(Nc\xi'/2)}{\cos(c\xi'/2)}\int_{-1}^{1}\frac{\tau^{n-1}\sin(Nc\xi/2)}{\cos(c\xi/2)}\frac{F(\xi)}{|P_L(\tau)|}d\xi - \right.$$

$$\left. -\frac{\sin(Nc\xi'/2)}{\cos(c\xi'/2)}\int_{-1}^{1}\frac{\tau^{n-1}\cos(Nc\xi/2)}{\cos(c\xi/2)}\frac{F(\xi)}{|P_L(\tau)|}d\xi\right] \equiv 0, \tag{12}$$

where d_{nm} are elements of matrix D of coefficients of the two-dimensional polynomial

$$R_{L-1}(\tau,\tau') = \frac{\mathrm{Im}\left(P_L(\tau)P_L^*(\tau')\right)}{\tau-\tau'} = \sum_{n,m=1}^{L} d_{nm}\tau^{n-1}(\tau')^{m-1} \tag{13}$$

Since the set of functions

$$\left\{\frac{\tau^{n-1}\cos(Nc\xi/2)}{\cos(c\xi/2)}\right\}_{n=1}^{n=L}, \left\{\frac{\tau^{n-1}\sin(Nc\xi/2)}{\cos(c\xi/2)}\right\}_{n=1}^{n=L} \tag{14}$$

is linearly independent for $\xi \in [-1,1]$, identity (12) is equivalent to the following systems of equations

$$\sum_{n=1}^{L} d_{nm}\int_{-1}^{1}\frac{\tau^{n-1}\sin(Nc\xi/2)}{\cos(c\xi/2)}\frac{F(\xi)}{|P_L(\tau)|}d\xi = 0 \tag{15a}$$

$$\sum_{n=1}^{L} d_{nm}\int_{-1}^{1}\frac{\tau^{n-1}\cos(Nc\xi/2)}{\cos(c\xi/2)}\frac{F(\xi)}{|P_L(\tau)|}d\xi = 0, \tag{15b}$$

m=1,2,...,L.

One can prove that

$$\det D = \frac{(-1)^{[L/2]}}{(2i)^L} \prod_{n,m=1}^{L} (\eta_n - \eta_m^*) \tag{16}$$

where square brackets denote the integer part of a quantity. The determinant does not equal zero owing to the properties of η_n. This means that equations (15) can be satisfied only if $\eta_n, n = 1,2,..., L$ solve the following set of transcendental equations

$$\int_{-1}^{1} \frac{\tau^n \sin(Nc\xi/2)}{\cos(c\xi/2)} \frac{F(\xi)}{|P_L(\tau)|} d\xi = 0, \tag{17a}$$

$$\int_{-1}^{1} \frac{\tau^n \cos(Nc\xi/2)}{\cos(c\xi/2)} \frac{F(\xi)}{|P_L(\tau)|} d\xi = 0, \tag{17b}$$

n=0,1,...L-1.

These equations are real. Global number of them is 2L, and it coincides with the number of the real quantities in the L complex unknown parameters η_n contained in the polynomial P_L. This means that equations (17) may be solvable at the fixed value of c. But solutions of (17) exist not always. There are some limitations on the relation between values of the parameters c and L which are necessary to be satisfied for existence of the solution. We establish here the main of these limitations.

Assume that equations (17a) are satisfied for n=0,1,...L-1. That means, that the function

$$\Phi_s(\xi) = \frac{\sin(Nc\xi/2)}{\cos(c\xi/2)} \frac{F(\xi)}{|P_L(\tau)|} \tag{18}$$

is orthogonal to any polynomial Q_m of degree less than L. This function must change its sign at the interval (-1,1) not less than L times. In other case one can construct the real polynomial Q_m of degree less than L whose zeros coincide with zeros of $\Phi_s(\xi)$. But function $\Phi_s(\xi)$ cannot be orthogonal to such a polynomial, because their product is real and it do not change its sign at the interval $(-1;1)$. From other side, the change of sign of $\Phi_s(\xi)$ can belong only to the factor *sin* $c\omega$, what gives the inequality

$$\frac{Nc}{2} > \left[\frac{L}{2}\right]\pi \tag{19a}$$

Similarly, from equations (17b) one can get the inequality

$$\frac{Nc}{2} > \left(\left[\frac{L+1}{2} \right] - \frac{1}{2} \right) \pi \tag{19b}$$

Both these inequalities together give

$$L < \frac{Nc}{\pi}. \tag{20}$$

This inequality shows that the number of unknown complex parameters η_n in the solutions of equation (4) is less than the general number N of unknown values I_n of the current at the antenna elements. In particular, the number of η_n is limited, when N increases but product Nc (this means, the antenna length) is constant. Note, that inequality (20) is independent of $F(\xi)$. In fact, for the concrete given functions $F(\xi)$ the maximal values of L are essentially smaller.

Solutions with different values of L can exist simultaneously at the same value of c . Moreover, different solutions can also exist at one of the same value of L. Indeed, in any solution $f(\xi)$ with parameters η_n, n = 1, 2, ..., L, one can substitute each parameter η_n by η_n^*. Such a substitution do not change function $|P_L(\tau)|$ and equations (17) remain to be satisfied. One can prove that this substitution do not also change the amplitude pattern $|f(\xi)|$. Hence, the value of functional σ is not changed, too. There exists a group of different solutions corresponding to the all possible combinations $\{\eta_n, \eta_m^*\}$, $n, m = 1, 2, \ldots, L$, $n \neq m$, which give one of the same amplitude pattern $|f(\xi)|$ and, consequently, one of the same value of σ. But these solutions differ one from others by the current distributions I_n calculated by formula (6). In what follows we will consider only one of the representatives from such a group of solutions, that has $\operatorname{Im}\eta_n > 0$, n = 1, 2, ..., L. Several different groups of such a type of solutions with the one of the same value of L can exist, too.

At some values of parameter c the transcendental equations (17) with different values of L can be satisfied simultaneously. These values of c are the branching points of solutions.

Note, that formula (8) can only describe solutions of equation (4) nonvanishing at the interval (-1; 1) and only such a type of solutions is considered here. Solutions which values in some points of the interval (-1,1) can equal zero are interesting for the case if the prescribed amplitude pattern $F(\xi)$ has a two-lobes form or it have be equal to zero in some fixed points. This case needs a special consideration. For the continual linear antenna it is partially considered in [11].

4. NUMERICAL RESULTS

As an example, the numerical results for the antenna array with 5 and 11 elements are presented and analyzed below. The prescribed amplitude pattern was taken in the form $F(\xi) \equiv 1$. Transcendental equations (17) were solved by the Newton method.

In Fig. 1 the dependence on *c* of the real and imaginary parts of parameters η_n are presented. The curves are marked by two digits *n, m* where *n* is the index at the parameters and *m* denotes the type of solution, joining the value of *L* and the solution number. We show only one solution from each equivalent group, that has positive imaginary part.

At the small values of *c* there are no solutions of equations (17) with *L*>0. This means that only patterns with $\arg f(\xi) \equiv 0$ satisfy eq. (4) at such values of *c*. At some $c = c_1$ an imaginary solution η_1 of (17) with *L*=1 arises, which gives rise the polynomial of the first degree $P_1(\tau) = (1 - \eta_1\tau)$. Value $c = c_1$ is the first branching point. (The characteristic points $c = c_m$ are marked on the figures for *N*=5.)

In the second branching point $c = c_2$ a new solution of (17) with imaginary $\eta_{1,2}, \eta_{2,2}$ arises with *L*=2, which gives rise the polynomial of the second degree

$$P_{2,1}(\tau) = (1 - \eta_{1,2}\tau)(1 - \eta_{2,2}\tau) = 1 + \eta_{1,2}\eta_{2,2}\tau^2 - \left(\eta_{1,2} + \eta_{1,2}\right)\tau$$

Solution η_1 remains to exist. Both these solutions give the odd phase patterns arg *f*(ξ) that corresponds to the real (but nonsymmetrical) current distributions [7].

After the value of $c = c_3$ at which $\eta_{1,2} = \eta_{2,2}$, the image solution $\eta_{1,2}, \eta_{2,2}$ transforms into the complex one with $\eta_{1,2} = -\eta_{2,2}^*$. It gives rise the polynomial

$$P_{2,2}(\tau) = (1 - \eta_{1,2}\tau)(1 + \eta_{1,2}^*\tau) = 1 + \left|\eta_{1,2}\right|^2\tau^2 - 2i\,\mathrm{Im}\,\eta_{1,2}\tau$$

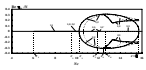
and also gives the odd phase pattern. As before, the current distribution is real and nonsymmetrical. But now the corresponding equivalent group also contains the solution with $\eta_{1,2} = -\eta_{2,2}$, which gives rise $P_{2,3}(\xi) = 1 - \eta_{1,2}^2\tau^2$. This solution gives the even phase pattern $\arg f(\xi)$ that corresponds to a complex symmetric current distribution. Both solutions with even and odd phase patterns give the same amplitude pattern and, following this, the same value of the functional σ. This fact was discovered numerically and described in details in [7]. Now it have got a new simple interpretation.

The solution with two complex $\eta_{1,2}, \eta_{2,2}$ exists up to $c = \pi$. From them the solution with L=3, having two complex parameters η_{13}, η_{23}: $\eta_{23} = -\eta_{13}^*$ and an imaginary one $\eta_{3,3}$, is branched off at some $c = c_4$. This solution gives rise the polynomial

$$P_3(\tau) = (1 - \eta_{1,3}\tau)(1 + \eta_{1,3}^*\tau)(1 - \eta_{3,3}\tau)$$

which gives the odd phase pattern. But other characteristic solution from this equivalent group has $\eta_{1,3} = -\eta_{2,3}$, $\eta_{3,3}$ and gives rise the polynomial which has not any phase symmetry.

Let us return to the imaginary solution η_1 after $c = c_1$. At some value of $c = c_5$ there arises a second solution of such a type and these two solutions exist simultaneously up to the value of $c = c_6$ where they coincide and disappear.


(a)

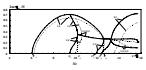

(b)

Figure 1. Solutions of equations (17) for array with 5 (solid lines) and 11 (dashed lines) elements; $F(\xi) \equiv 1$.

All solutions in each of above mentioned groups give one of the same value of functional σ in the group, but these values are, in general, different for the different groups. Recall, that different solutions of equations (17) and, respectively, of equation (4) can give different values of the functional, from which only global-minimum value is interesting physically. To choose the optimal solutions, which are the global-minimum points of the functional, the values of σ corresponding to different groups of solutions should be compared. In Fig. 2 the value of σ at the equiphase solution $\arg f(\xi) \equiv 0$ and at the optimal one are given for N=5. Digits at the curves are equal the values of parameter L in the optimal solution. As rule, the optimal solution corresponds to the maximal value of L at the fixed c. But this rule is disturbed after $c > c_7$ when the solution with L=2 is optimal although the solution with L=3 continues to exist.

The fact that value $\sigma = 0$ reaches at $\boldsymbol{c}$=π is explained by the special form of the desired amplitude pattern $F(\xi)$, which is realizable exactly in this case (e.g. by the current distribution $I_n = \delta_{0n}$).

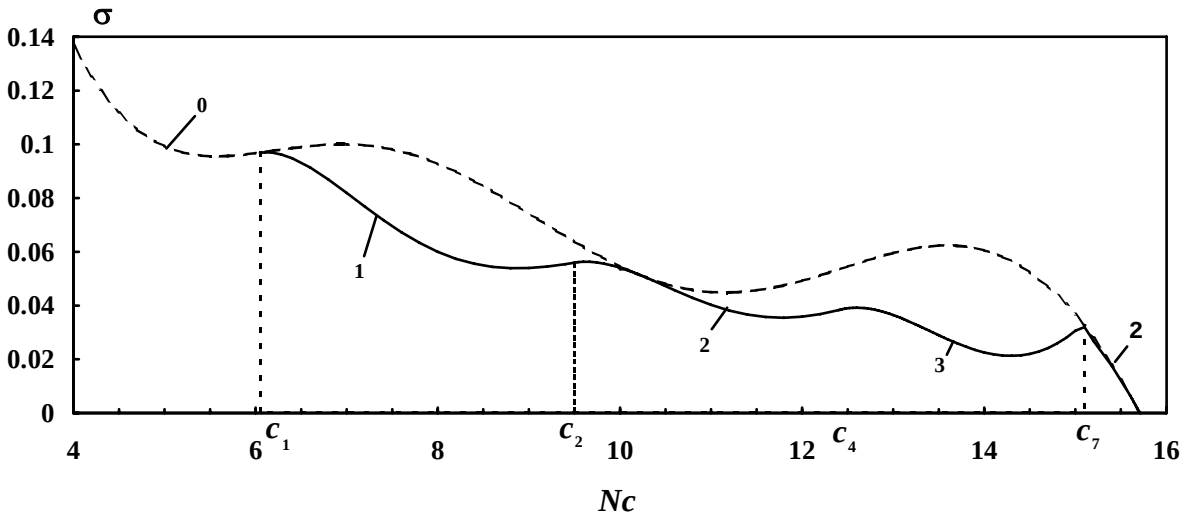


Figure 2. Value of functional σ at the optimal (solid line) and equiphase (dashed line) solutions of equations (17) for array with 5 elements; $F(\xi) \equiv 1$.

5. CONCLUSION

The nonlinear integral equation describing the solutions of the problem of synthesis of the equidistant linear antenna array according to the prescribed amplitude directivity pattern, is solved analytically with some number of the complex parameters which are calculated from a system of transcendental equations. It is shown that the number of such parameters are limited and it is smaller than the number of radiators in the array. There exist the equivalent groups of solutions which have one of the same amplitude directivity pattern but different phase ones. Correspondingly, these patterns are formed by the different current distributions. The solutions are branched out at some values of the physical parameters where solutions of new types are arisen. Numerical results show that the optimal solution can be chosen among the different equivalent groups.

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