AN OPTIMAL SHAPE OF A LOOP, MOVING IN A MAGNETIC FIELD WITH A NON-SINUSOIDAL STRUCTURE, ENSURING MAXIMUM EMF

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Abstract

A method for determination of the optimal shape of a loop, moving in a magnetic field with a periodic non-sinusoidal structure, is presented. The optimum criterion is the maximum of the induced electromotive force (EMF). The equivalent loop shape is computed by an iterative solution of an isoperimetric variational problem. The necessary conditions for an extremum are introduced by four-dimensional (4-D) electromagnetic potentials. Magnetic fields with different non-sinusoidal structure are investigated.

1. INTRODUCTION

A relative translation movement of a closed loop in a magnetic field with a periodic non-sinusoidal structure is considered. The optimal shape of a loop *1,* which is related to the fixed coordinate system XYZ (Fig.1) and is situated in a plane parallel to YZ, has to be found. The magnetic field is connected with the moving coordinate system X'Y'Z'. The magnetic flux density B' (or simply magnetic field) is X'-directed and it is constant at Y'-direction while it has a non-sinusoidal piecewise-linear profile at Z'-direction, as shown in Fig.1.

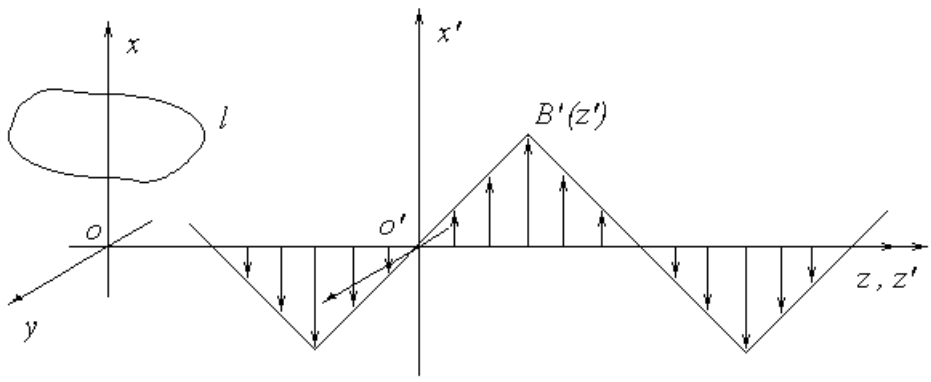

Figure 1. The form of the non-sinusoidal magnetic field*:*

l - optimal shape of the loop , *B'* - magnetic field

The magnetic field satisfies the Dirichlet condition. As a consequence of its periodicity, it can be represented by the following Fourier series

$$B'(z') = \sum_{k=1}^{n\to\infty} B_{km} \sin\frac{\pi}{\tau} z'_k \ ,$$

(1)

where B_{km} is the amplitude of the *k*-th harmonic of the magnetic field, τ is a the step-size in z'. By using Fourier analysis with B_1 = 1T , $\frac{\pi}{\tau}$ = 0.78 m^{-1} , we have obtained the following approximation, retaining the odd harmonics up to ninth order

$$\begin{aligned} B'(z') = {} & 0.81\sin 0.78z' - 0.09\sin 2.34z' + 0.03\sin 3.9z' \\ & - 0.02\sin 5.46z' + 0.01\sin 7.02z' \end{aligned}$$

(2)

2. FOUR-DIMENSIONAL ELECTROMAGNETIC POTENTIAL

The components of the vector induced electrical field E in the loop, as a result of the moving magnetic field, are obtained by means of a 4-D electromagnetic potential in the Minkovski space. The procedure here follows [4]. In this paper a 4-D electromagnetic potential is considered in the form

$$\bar{\Psi}_{(\mu)} = \left\{\Psi_{1\mu}, \Psi_{2\mu}, \Psi_{3\mu}, \Psi_{4\mu}\right\} = \left\{A_{\mu x}, A_{\mu y} A_{\mu z}, \frac{j}{c} V_{\varepsilon}\right\}$$

(3)

A magnetic vector potential is used because of the fact that the magnetic field is created by a source current [2,6]. The Lorentz transformations of the electromagnetic potentials are [1,2,4]

$$A_{\mu x} = A'_{\mu x'} \ ; A'_{\mu y} = A'_{\mu y'} \ ; A'_{\mu z} = \alpha\left(A'_{\mu z'} + \frac{v}{c^2} V'_{\varepsilon}\right); V_{\varepsilon} = \alpha\left(V'_{\varepsilon} + vA'_{\mu z'}\right)$$

(4)

as well as Lorentz transformation of the coordinates are

$$x' = x\,,\; y' = y\,,\; z' = \alpha(z - vt)\,,\; t' = \alpha\left(t - \frac{v}{c^2}z\right),$$

(5)

The components of the 4-D potential in the moving coordinate system are

$$\vec{\Psi}_\mu = \begin{cases} \Psi_1 = A_{\mu x} = A'_{\mu x'} = o \\ \Psi_2 = A_{\mu y} = A'_{\mu y'} = -\dfrac{\tau}{2\pi}\sum\limits_{k=1}^{n} B_{km}\cos\dfrac{\pi}{\tau}\alpha(z - vt) \\ \Psi_3 = A_{\mu z} = \alpha\left(A'_{\mu z'} + \dfrac{v}{c^2}V'_\varepsilon\right) = \dfrac{1}{2}y\alpha\sum\limits_{k=1}^{n} B_{km}\sin\dfrac{\pi}{\tau}\alpha(z - kvt) \\ \Psi_4 = \dfrac{j}{c}V_\varepsilon = \alpha\left(V'_\varepsilon + vA'_{\mu z'}\right) = \dfrac{j}{c}\dfrac{1}{2}\alpha vy\sum\limits_{k=1}^{n} B_{km}\sin\dfrac{\pi}{\tau}\alpha(z - kvt) \end{cases},$$

(6)

where $\alpha = \dfrac{1}{\sqrt{1 - v^2/c^2}}$ is the relativity factor and c is the free-space *light velocity* ($c \approx 3\times 10^8\, m/s$).

3. INDUCED ELECTRICAL FIELD

The components of the electrical field can be associated with the 4-D Maxwell tensor F [4-6]. It can be derived from the 4-D potential using the following derivatives

$$F_{ik}^{(\varepsilon)} = \frac{\partial \Psi_k}{\partial x_j} - \frac{\partial \Psi_j}{\partial x_k} \quad ,(j,k = 1,2,3,4),$$

(7)

where $\vec{X} = \{x, y, z, jct\}$ are the 4-D coordinates in the Minkovski space, j is the imaginary unit. For the components of the electrical field, induced in the loop, from (1) and (7) follows

$$E_x = 0$$

$$E_y = \frac{1}{2}\frac{\alpha^2 \pi v}{\tau}\sum_{k=1}^{n} kB_{km}\sin\frac{\pi}{\tau}\alpha(z - kvt) - \frac{1}{2}\alpha v\sum_{k=1}^{n} B_{km}\sin\frac{\pi}{\tau}\alpha(z - kvt)$$

(8)

$$E_z = \frac{1}{2}\frac{\alpha^3\pi^2 vy}{\tau^2}\sum_{k=1}^{n} kB_{km}\cos\frac{\pi}{\tau}\alpha(z-kvt) - \frac{1}{2}\frac{\alpha^3\pi^2 vy}{\tau^2}\sum_{k=1}^{n} B_{km}\cos\frac{\pi}{\tau}\alpha(z-kvt)$$

4. NECESSARY CONDITIONS FOR AN EXTREMUM

The maximum induced EMF is obtained from the extremum condition of the following functional [2]

$$U_1[y(x,z)] = e(t) = \int_{(l)} \vec{E}d\vec{l} \to \max \ for \ \forall \ t_1$$

(9)

which in a parametric form is

$$U_1[y(x,z)] = e(t) = \int_{\tau_0}^{\tau_1} \left(E_x\dot{x} + E_y\dot{y} + E_z\dot{z}\right)d\tau \to \max \ for \ \forall \ t_1$$

(10)

It gives for a loop, situated in the YZ - plane the expression

$$U_1[y(x,z)] = e(t) = \int_{\tau_0}^{\tau_1} \left(E_x\dot{x} + +E_z\dot{z}\right)d\tau \to \max \ for \ \forall \ t_1 \ ,$$

(11)

where $\dot{x}, \dot{y}, \dot{z}$ are the derivates with respect to the parameter τ and the values of τ_0 and τ_1 are related to the beginning and the end of the loop.

The functional extremum has to be found at a presence of an additional restriction for the loop perimeter L - this is an isoperimetric problem

$$U_2[y(x,z)] = \int_{\tau_o}^{\tau_1} \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)d\tau = L \ , L = const$$

(12)

The necessary conditions for an extremum have to be found on the basis of the Langrange function [2]

$$L[y(x,z), \lambda\ ] = \left(E_x\dot{x} + E_y\dot{y} + E_z\dot{z}\right) + \lambda\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)^{1/2}$$

(13)

where λ is the Langrange constant.

The necessary conditions for an extremum are derived as a system of differential equations (see the A*ppendix*). The equations are obtained on the basis of the Euler equations, applying for the loop in a parametric form. The optimal loop's shape has to be found in a plane, parallel to YZ-plane. The additional conditions $\dot{x} = 0$, $\ddot{x} = 0$ yield

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = \lambda \frac{\ddot{y}\dot{z} - \dot{y}\ddot{z}}{\left(\dot{y}^2 + \dot{z}^2\right)^{3/2}}$$

(14)

where $\dot{y}$, $\ddot{y}$, $\dot{z}$, $\ddot{z}$ are the derivatives with respect to the parameter τ.

The following an explicit equation is obtained

$$-\frac{\lambda\, \ddot{y}(z)}{\left[1 + \dot{y}(z)^2\right]^{3/2}} = v\pi\left(\frac{\pi}{\tau} - 1\right)\sum_{k=1}^{\infty} B_{km} \cos\frac{\pi}{\tau}(z - kvt) -$$

$$- v\frac{\pi^2}{\tau^2}\sum kB_{km} \cos\frac{\pi}{\tau}(z - kvt)$$

(15)

The solution of equation (15) is a necessary condition for an optimal loop's shape finding.

5. SUFFICIENT CONDITIONS FOR AN EXTREMUM

The solution of equation (15) is a solution of a Cauchy problem with initial conditions: $y(z)_{z=0} = y_1$, $\dot{y}(z)_{z=0} = \dot{y}_1 = 0$.

The curvature of the extremal curve is determined from the right part of equation (15). The Lagrange constant is defined with respect of the loop's parameter. By numerical experiments is found that for different values of λ, using the initial value problem (transverse conditions), the same central extremal field is obtained.

If the Legendre conditions $\frac{\partial^2 L}{\partial y \partial y} < 0$, $\frac{\partial^3 L}{\partial y \partial y \partial y} \neq 0$ are fulfilled, then the extremum of the functional is guaranteed.

For values of $\lambda < 0$ a maximum value of the EMF is ensured.

The fulfillment of the sufficient condition (Jacoby condition) is verified by numerical experiments.

6. NUMERICAL RESULTS AND DISCUSSION

A family of characteristics for the parameters $\frac{\pi}{\tau} = 0.78m^{-1}$, $v_1 = 402\, m/s$, $\lambda = -0.5$ at different moments $t_1\left(t_1 = 0, T/8, T/4, 3/8\,T, T/2, 5/8\,T, 7/8\,T, T\right)$ is obtained. The numerical solution shows the periodic structure of the extremals. Some of the extremals are shown in

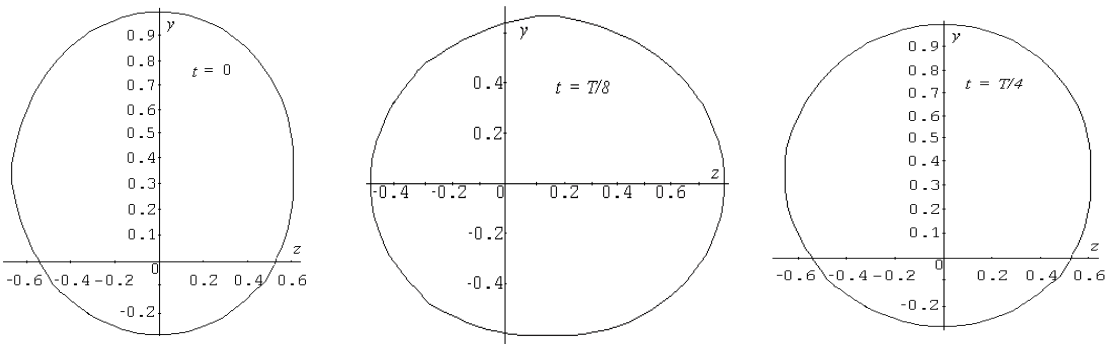

Figure 2.

Figure 2. Optimal loop's shapes for a magnetic field , presented in Fig.1, at different moments t_1

The obtained results show the presence of a periodicity (in time) of the computed extremals. Since the extremals curvature changes in time an equivalent form of the optimum loop have to be introduced. The equivalent form of the optimum loop presented in Figure.3 is obtained on the extremals at different moments t_1 .

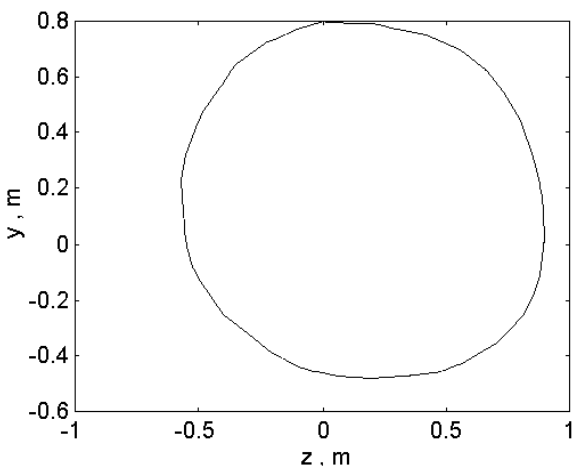

Figure 3. An equivalent optimal loop's shape

for a magnetic field, presented in Figure 1.

The equivalent form of the optimal loop computed from the solution of the variational problem is a smooth and reminds a circle.

In order to determine the influence of the other parameters on the shape of the optimal loop, numerical experiments are done.

The results obtained for a high velocity v (close to c), are of interest. In Figure 4. the optimal shape of the loop, ensuring maximum EMF for a sufficiently high velocity $v = 18000\ m/s$, is presented.

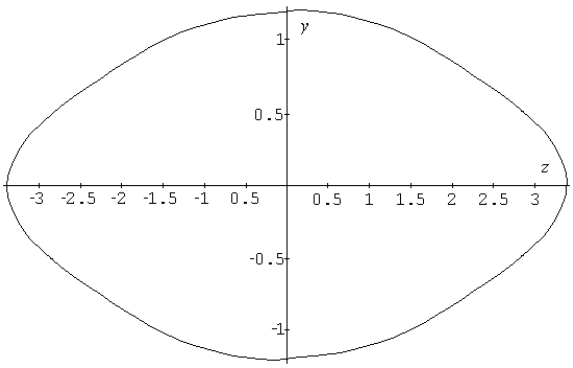

Figure 4. An optimal loop's shape for a velocity $v = 18000 m/s$.

Fig.4. shows a deformation of the loop's shape in the direction of the movement.

Investigations of other forms of the traveling magnetic field are also done. Two more forms - the trapezoidal form, presented in Fig.5, and also the ideal sinusoidal form, are investigated.

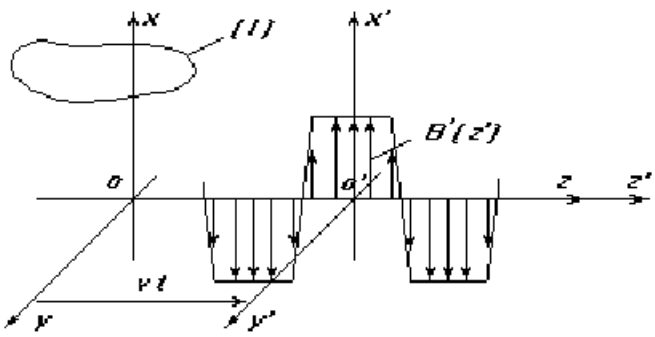

Figure 5. A trapezoidal non-sinusoidal form of the magnetic field.

The optimal form for a sinusoidal field is very close to a circle.

Figure 6 presents optimal forms at different moments when a trapezoidal magnetic field is used.

The equivalent form of the optimal loop represented in Fig.6, is shown in Fig.7.

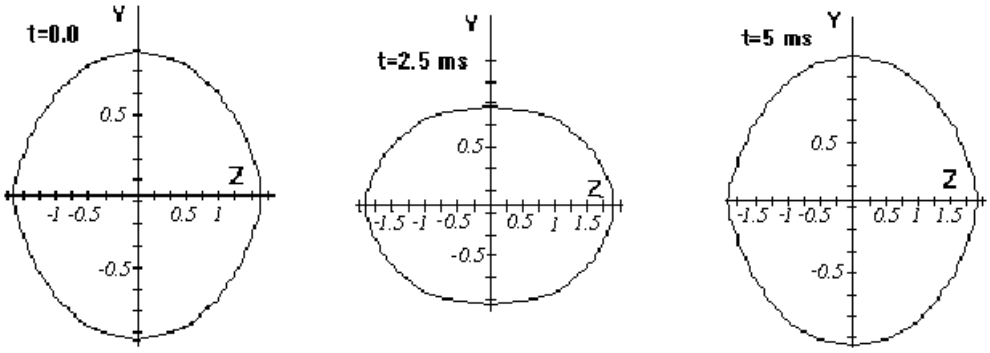

Figure 6. Optimal loop's shapes for a magnetic field, shown in Figure 5, at different moments.

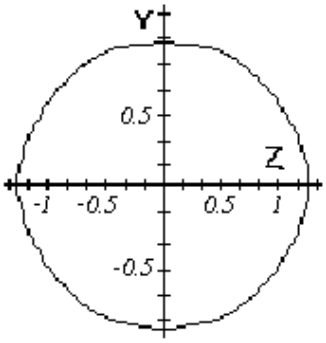

Figure 7. An equivalent optimal loop's shape for a magnetic field, shown in Figure 5.

7.CONCLUSION

The results obtained in this paper enable to determine the optimal form of loop, ensuring maximum EMF, which is induced by a moving periodic magnetic field, with a different non-sinusoidal structure.

The algorithm developed here is based on a solution of a variational problem, using four-dimensional potentials in the Minkovski space. The investigations are valid for different velocities.

The numerical results obtained in the paper show that for a low velocity ($v \langle\langle c$) the

optimal form of the loop is an invariant with respect to the velocity. For a relativistic velocity ($v \approx c$) the optimal form of the loop is strongly deformed.

For a sinusoidal magnetic field the equivalent optimal form is very close to a circle. For a non-sinusoidal magnetic field the optimal form of the loop differs from a circle as it can be seen in Fig.2 and in Fig.6.

The results obtained here are valid for a time-harmonic magnetic field. The investigations may be continued also for other types magnetic fields, using again the criterion of a maximum EMF.

The problem of determination of the optimal loop's shape, ensuring maximum EMF, is in use for electromagnetic systems working with a non-sinusoidal traveling magnetic field, by a condition that the velocity is low comparing to c. Such systems are the MHD-generators, linear induction motors, liquid metal pumps, etc. Examples for a high velocity are the electromagnetic systems for magnetic resonance imaging, electromagnetic guns, etc. The approach proposed here may be also used in the case of electromagnetic systems with a rotation.

The obtained results are valid for linear systems, but they can be used in combination with other methods for optimization of non-linear systems also.

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APPENDIX

$$\left|\begin{array}{l}
\dot{y}\left(\dfrac{\partial A_{\mu y}}{\partial x}-\dfrac{\partial A_{\mu x}}{\partial y}\right)-\dot{z}\left(\dfrac{\partial A_{\mu x}}{\partial z}-\dfrac{\partial A_{\mu z}}{\partial x}\right)=\lambda\dfrac{\ddot{x}\left(\dot{y}^2+\dot{z}^2\right)-\dot{x}\left(\ddot{z}\dot{z}+\ddot{y}\dot{y}\right)}{\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)^{3/2}}\\
\dot{z}\left(\dfrac{\partial A_{\mu z}}{\partial y}-\dfrac{\partial A_{\mu y}}{\partial z}\right)-\dot{x}\left(\dfrac{\partial A_{\mu y}}{\partial x}-\dfrac{\partial A_{\mu x}}{\partial y}\right)=\lambda\dfrac{\ddot{y}\left(\dot{z}^2+\dot{x}^2\right)-\dot{y}\left(\ddot{z}\dot{z}+\ddot{x}\dot{x}\right)}{\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)^{3/2}}\\
\dot{x}\left(\dfrac{\partial A_{\mu x}}{\partial z}-\dfrac{\partial A_{\mu z}}{\partial x}\right)-\dot{y}\left(\dfrac{\partial A_{\mu z}}{\partial y}-\dfrac{\partial A_{\mu y}}{\partial z}\right)=\lambda\dfrac{\ddot{z}\left(\dot{y}^2+\dot{x}^2\right)-\dot{z}\left(\ddot{x}\dot{x}+\ddot{y}\dot{y}\right)}{\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)^{3/2}}
\end{array}\right.$$