APPLICATION OF KUMMER FUNCTIONS IN THE ANALYSIS OF SLOW WAVES IN THE CIRCULAR GYROTROPIC WAVEGUIDE

Georgi N. Georgiev*, Nikolaos K. Uzunoglu** and Mariana N. Georgieva-Grosse***

* Faculty of Mathematics and Informatics, University of Veliko Tirnovo "St. St. Cyril and Methodius", BG-5000 Veliko Tirnovo, Bulgaria
e-mail: gngeorgiev@yahoo.com

** Institute of Communication and Computer Systems, National Technical University of Athens, 9, Iroon Polytechniou Str., GR-15773 Zografos, Athens, Greece
e-mail: nuzu@central.ntua.gr

*** Meterstrasse 4/2, D-70839 Gerlingen, Germany
e-mail: Mariana.G@t-online.de

Abstract

A lemma concerning an important property of the real zeros of Kummer function is proved numerically. Based on it and on some known attributes of the latter, criteria for slow TE_{0n} modes propagation in a circular waveguide with azimuthally magnetized ferrite are drawn. The phase characteristics of the configuration for slow TE_{01} mode are presented.

1. INTRODUCTION

The exact methods for analysis of the circular ferrite or solid-plasma waveguides with azimuthal magnetization, under TE_{0n}, resp. TM_{0n} modes excitation are based on the confluent hypergeometric functions (CHFs) [1]-[11] or on the Bolle-Heller functions [12]-[17]. As pointed out by L.J. Slater [18], confluent hypergeometric are called four functions: the Kummer and the connected with it Tricomi function $\Phi(a,c;x)$ and $\Psi(a,c;x)$, resp. and the two Whittaker functions $M_{\kappa,\mu}(x)$ and $W_{\kappa,\mu}(x)$. One faces the problem of multiple-valuedness when handling with $M_{\kappa,\mu}(x)$ as well as with $W_{\kappa,\mu}(x)$ [5], [18], [19] though on the other hand their symmetry [5], [19] is an important advantage in the applications [5]. The Tricomi CHF is also multiple-valued [5]-[7], [18], [20], [21]. Both $\Phi(a,c;x)$ and $\Psi(a,c;x)$ are not symmetrical [5]-[7]. In our opinion however, the Kummer function is to be preferred with respect to the Whittaker first function in the investigation of guiding lines, containing circular gyrotropic layer of single-connected cross-section, including the point zero, just in view of its single-valuedness [5]-[8], [18]. Not knowing the behaviour of CHFs in the complex field, Bolle and Heller considered that it seemed more convenient to replace them by the appropriately constructed real functions

$B_1(\alpha;x)$ and $H_1(\alpha;x)$ [12]. Subsequently Ince and Tsandoulas [13], Bernues and Bolle [14] and Samaddar [17] redefined them. Their modified counterparts $P_1(\alpha;x)$ and $Q_1(\alpha;x)$ have also been introduced [13], [14] to range over various aspects of the propagation phenomena in the structures treated.

The complex Kummer CHFs have been used to study normal TE_{0n} modes in the azimuthally magnetized circular ferrite waveguides [5]-[10]. Exhibiting phase shifting properties, the latter are attractive for phase shifter elements [5], [7], [8], [11]-[15]. Graphs have been obtained, presenting the dependence of differential phase shift they produce on frequency [7], [8]. A significant step forward was the recent substantiation of formulae for direct computation of this important quantity from the parameters of the ferrite waveguide, using a specially elaborated iterative technique [11].

Unlike the normal TE_{0n}, resp. TM_{0n} modes, the behaviour of slow ones in the class of structures examined, is not known. The possibility for their existence has only been mentioned in a few works [1], [3], [14]. An exception is the study of slow surface waves in the solid-plasma coaxial line by Obunai and Hakamada [4]. The introduction of the real Kummer CHFs as wave functions for propagation, made here, reveals that slow TE_{0n} modes may also be transmitted in the uniform in radial direction gyrotropic waveguide for negative magnetization of the medium. Using an important result for the real positive zeros of wave function, established numerically, in combination with its other properties [18], [20], [21], the propagation conditions are determined. Following the approach, developed in [8], the phase curves for TE_{01} mode are computed and investigated. It is proved that the symmetry is their inherent characteristic in both areas in which the wave might be sustained.

2. A PROPERTY OF THE REAL ZEROS OF KUMMER FUNCTION

Tricomi [20], [21] has proved that if $\hat{a}, \hat{c}, \hat{x}$ are real, $\hat{x} > 0$ and $\hat{c} > 0$: i) The Kummer CHF $\Phi(\hat{a}, \hat{c}; \hat{x})$ [18], [20], [21] has real positive zeros only if $\hat{a} < 0$. ii) The number of zeros $l = -[\hat{a}]$ is finite, ($[\hat{a}]$ is the largest integer less or equal to $\hat{a}$, i.e. $[\hat{a}] \le \hat{a}$). iii) At the point $\hat{a} = [\hat{a}] = -(n-1)$, ($n, l$ - positive integers, $n \le l$) a new zero appears. (Note that provided $\hat{a} \ge 0$, $l = 0$.)

Let $\hat{k} = \hat{a} - \hat{c}/2$ and $\hat{\varsigma}_{\hat{k},n}^{(\hat{c})}$ $(n = 1,2,...,l)$ be the consecutive zeros of Φ in $\hat{x}$ for specific $\hat{c}$. The numerical analysis shows that it holds $\lim\limits_{\substack{\hat{k} \to -(n-1)-\hat{c}/2 \\ \hat{k} < -(n-1)-\hat{c}/2}} \hat{\varsigma}_{\hat{k},n}^{(\hat{c})} = +\infty$ and $\lim\limits_{\hat{k} \to -\infty} \hat{\varsigma}_{\hat{k},n}^{(\hat{c})} = 0$. Besides, it turns out that the products $|\hat{k}| \hat{\varsigma}_{\hat{k},n}^{(\hat{c})}$ are of special interest if $|\hat{k}|$ becomes very large (cf. Table 1). The numerical results permit to formulate the following Lemma.

Lemma 1: If $\hat{\varsigma}_{\hat{k},n}^{(\hat{c})}$ is the nth real positive zero of the Kummer CHF $\Phi(\hat{a}, \hat{c}; \hat{x})$ in case $\hat{a}, \hat{c}, \hat{x}$ - real, $\hat{x} > 0$, $\hat{c} > 0$, restricted and $\hat{k} = \hat{a} - \hat{c}/2$, the infinite sequence of numbers $\{|\hat{k}| \hat{\varsigma}_{\hat{k},n}^{(\hat{c})}\}$ is convergent for $\hat{k} \to -\infty$ and its limit $\hat{L}$ depends on $\hat{c}$ and n.

Table 1. Values of the first real positive zeros $\hat{\varsigma}_{\hat{k},1}^{(3)}$ of $\Phi(1.5 + \hat{k}, 3; \hat{x})$ and of the products $|\hat{k}| \hat{\varsigma}_{\hat{k},1}^{(3)}$ for large negative $\hat{k}$.

$\hat{k}$	$\hat{\varsigma}_{\hat{k},1}^{(3)}$	$\lvert\hat{k}\rvert \hat{\varsigma}_{\hat{k},1}^{(3)}$	$\hat{k}$	$\hat{\varsigma}_{\hat{k},1}^{(3)}$	$\lvert\hat{k}\rvert \hat{\varsigma}_{\hat{k},1}^{(3)}$
- 1000	0.0065936585	**6.5936**585	- 10000	0.0006593654151	**6.593654**151
- 2000	0.0032968276	**6.5936**552	- 20000	0.0003296827058	**6.593654**117
- 4000	0.0016484135	**6.59365**43	- 40000	0.0001648413527	**6.5936541**09
- 6000	0.0010989423	**6.59365**42	- 60000	0.0001098942351	**6.5936541**08
- 8000	0.00082420677	**6.5936541**	- 80000	0.00008242067634	**6.593654107**
-10000	0.00065936541	**6.5936541**	-100000	0.00006593654107	**6.593654107**

In particular if $\hat{c} = 3$, $n = 1$, then $\hat{L}(\hat{c}, n) = \hat{L}(3,1) = 6.593654107$. The figures in the products, listed in the third and sixth columns of Table 1 as a function of $\hat{k}$ which

coincide with the limit $\hat{L}(3,1)$, are distinguished by bold-face. Their increasing number with growing $\hat{k}$ confirms the statement of Lemma 1.

3. APPLICATION TO AZIMUTHALLY MAGNETIZED CIRCULAR FERRITE WAVEGUIDE ANALYSIS

The solution of propagation problem for slow TE_{0n} modes in a circular waveguide with ferrite, magnetized azimuthally to remanence by an infinitely thin central conductor yields the equation

$$\Phi(\hat{a},\hat{c};\hat{x}_0)=0\,. \tag{1}$$

The ferrite has a permeability tensor with off-diagonal element $\alpha = \gamma M_r / \omega$ (γ - gyromagnetic ratio, M_r - remanent magnetization, ω - angular frequency of the wave) and a scalar permittivity $\varepsilon = \varepsilon_0 \varepsilon_r$. In eqn. (1) $\hat{a} = \hat{c}/2 + \hat{k}$, $\hat{c}=3$, $\hat{k} = \alpha\bar{\hat{\beta}} / \left(2\bar{\hat{\beta}}_2\right)$, $\bar{\hat{\beta}}_2 = \left[\bar{\hat{\beta}}^2 - \left(1-\alpha^2\right)\right]^{1/2}$, $\hat{x}_0 = 2\bar{\hat{\beta}}_2 \bar{r}_0$. The quantities $\bar{\hat{\beta}} = \hat{\beta} / \left(\beta_0 \sqrt{\varepsilon_r}\right)$, $\bar{\hat{\beta}}_2 = \hat{\beta}_2 / \left(\beta_0 \sqrt{\varepsilon_r}\right)$, $\bar{r}_0 = \beta_0 r_0 \sqrt{\varepsilon_r}$ are the phase constant, radial wavenumber and guide radius, normalized with the free space phase constant $\beta_0 = \omega\sqrt{\varepsilon_0 \mu_0}$ and relative permittivity ε_r. Eqn. (1) is satisfied if $\bar{\hat{\beta}}_2 = \hat{\varsigma}_{\hat{k},n}^{(\hat{c})} / \left(2\bar{r}_0\right)$. Accordingly, a finite number n of slow TE_{0n} modes, equal to that of the zeros $\hat{\varsigma}_{\hat{k},n}^{(\hat{c})}$ of $\Phi(\hat{a},\hat{c};\hat{x}_0)$ for relevant $\hat{a}$ may be sustained for $\alpha < 0$ only, since $\operatorname{sgn}\alpha = \operatorname{sgn}\hat{k}$ and $\hat{\varsigma}_{\hat{k},n}^{(\hat{c})}$ spring up provided $\hat{k} < -(n-1) - \hat{c}/2 < 0$, $(n \geq 1)$.

If σ is any of parameters α or $\bar{\hat{\beta}}$, it satisfies a biquadratic equation

$$\sigma^4 - \left(1 + \bar{\hat{\beta}}_2^2\right)\sigma^2 + 4\bar{\hat{\beta}}_2^2 \hat{k}^2 = 0\,, \tag{2}$$

with roots $\sigma_{1,2}$:

$$\sigma_{1,2}^2 = 0.5\left[\left(1 + \bar{\hat{\beta}}_2^2\right) \pm \sqrt{\left(1 + \bar{\hat{\beta}}_2^2\right)^2 - 4 \times 4\bar{\hat{\beta}}_2^2 \hat{k}^2}\,\right]. \tag{3}$$

(The sign + corresponds to σ_1 and the sign – to σ_2.) Only the positive solutions are taken into account. Denoting by $|\alpha_1|$, $|\alpha_2|$ and $\bar{\hat{\beta}}_{(1)}$, $\bar{\hat{\beta}}_{(2)}$ the roots of respective equations for α and $\bar{\hat{\beta}}$, it may be shown that

$$|\alpha_1| = \bar{\hat{\beta}}_{(2)}, \tag{4}$$

$$|\alpha_2| = \bar{\hat{\beta}}_{(1)}. \tag{5}$$

The numerical analysis indicates that σ_1 and σ_2 are real (the discriminant Δ = $\left\{\left(4\bar{r}_0^2 + \left[\hat{\varsigma}_{\hat{k},n}^{(\hat{c})}\right]^2\right)^2 - \left(8\bar{r}_0\,\hat{\varsigma}_{\hat{k},n}^{(\hat{c})}\left|\hat{k}\right|\right)^2\right\} / \left(4\bar{r}_0^2\right)^2 \geq 0$) in two regions, determined by the set of inequalities:

$$-(n-1) - \hat{c}/2 > \hat{k} \geq \hat{k}_{\lim}^{i}(\bar{r}_0), \tag{6}$$

$$\hat{k}_{\lim}^{ii}(\bar{r}_0) \geq \hat{k} > -\infty. \tag{7}$$

Thus, there are two areas in which slow TE_{0n} modes may be sustained. Relations (6) and (7) are the mathematical criteria for propagation. The points $\hat{k} = \hat{k}_{\lim}^{i}(\bar{r}_0)$ and $\hat{k} = \hat{k}_{\lim}^{ii}(\bar{r}_0)$, specifying the left, resp. right limit of the intervals of variation of parameter $\hat{k}$ depend on $\bar{r}_0$. If $\bar{r}_0 \to 0$ $(+\infty)$, $\hat{k}_{\lim}^{i}(\bar{r}_0)$ tends to $-\infty$ $[-(n-1) - \hat{c}/2$] and the first range expands (shrinks), e.g. assuming $\bar{r}_0$ = 0.02, 0.1 and 1.0, we have $\hat{k}_{\lim}^{i}(\bar{r}_0)$ = –6.477368, –3.004369 and –1.504732, resp. As for the second one, it widens provided $\bar{r}_0$ grows, for example in case $\bar{r}_0$ = 13.3, 15 and 50, then $\hat{k}_{\lim}^{ii}(\bar{r}_0)$ = –8.540894, –2.418563 and –1.500073, resp.

Since $\lim\limits_{\substack{\hat{k} \to -(n-1)-\hat{c}/2 \\ \hat{k} < -(n-1)-\hat{c}/2}} \sigma_1 \to +\infty$ and $\lim\limits_{\substack{\hat{k} \to -(n-1)-\hat{c}/2 \\ \hat{k} < -(n-1)-\hat{c}/2}} \sigma_2 \to 2\left|\hat{k}\right| = 2(n-1) + \hat{c}$, the condition for TE_{0n} mode transmission in the first area is

$$|\alpha| > 2(n-1) + \hat{c}, \tag{8}$$

($|\alpha|>3$ for TE_{01} mode). This corresponds to the case of strong anisotropy. Neglecting $[\hat{\varsigma}_{\hat{k},n}^{(\hat{c})}]^2$ in Δ when $\hat{k}\to-\infty$ and applying Lemma 1, the requirement for propagation in the second region is obtained:

$$\bar{r}_0 \geq 2\,\hat{L}(\hat{c},n). \tag{9}$$

As $\lim\limits_{\substack{\hat{k}\to-\infty \\ \bar{r}_0\to+\infty}} \sigma_1 \to 1$ and $\lim\limits_{\substack{\hat{k}\to-\infty \\ \bar{r}_0\to+\infty}} \sigma_2 \to 0$, we get

$$0<|\alpha|<1. \tag{10}$$

Consequently, now we are dealing with the case of weak anisotropy. The relations (8)-(10) constitute the physical criteria slow waves to be supported. The conditions (9) and (10) for the second area complement each other and should be employed simultaneously.

4. PHASE BEHAVIOUR

The dependence of $\bar{\hat{\beta}}$ on $|\alpha|$ with $\bar{r}_0$ as parameter for slow TE_{01} mode, computed from formula (3) in which the roots $\hat{\varsigma}_{\hat{k},n}^{(\hat{c})}$ of eqn. (1) are introduced, is plotted in Figs. 1, 2 for $|\alpha|>3$ and $0<|\alpha|<1$, respectively. It follows from eqns. (4), (5) that the $\bar{\hat{\beta}}(|\alpha|)$ phase curves in both areas are symmetric with respect to the bisectrix $\bar{\hat{\beta}}=|\alpha|$

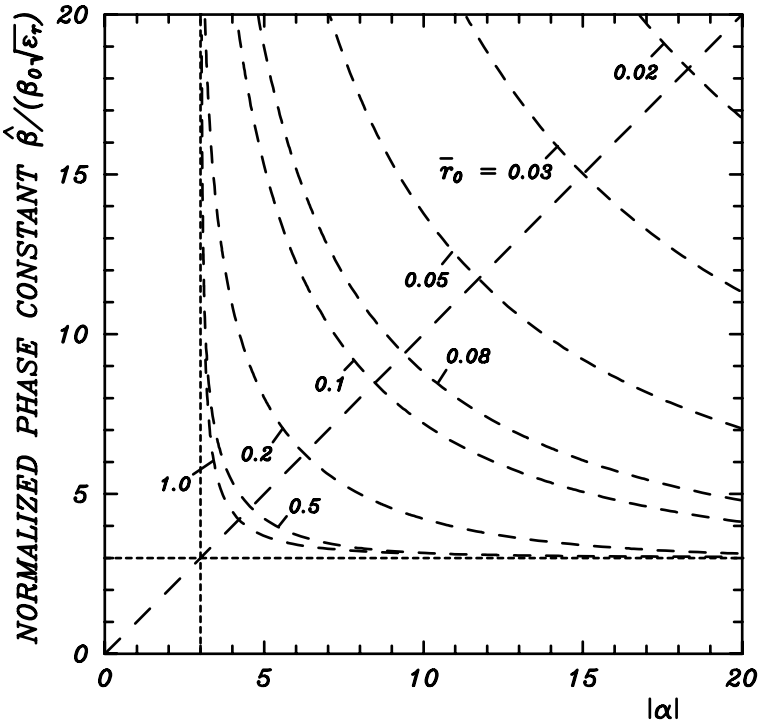

Figure 1. $\bar{\hat{\beta}}(|\alpha|)$ curves of slow TE_{01} mode for $|\alpha| > 3$.

of the right angle $|\alpha|\,0\,\bar{\hat{\beta}}$. The points at it correspond to the limiting values $\hat{k} = \hat{k}^{i}_{\lim}(\bar{r}_0)$, resp. $\hat{k} = \hat{k}^{ii}_{\lim}(\bar{r}_0)$ which means that for them $\Delta = 0$ and $\sigma^{i,ii}_{1\lim} = \sigma^{i,ii}_{2\lim} = \sqrt{0.5\left(1+\bar{\hat{\beta}}_2^2\right)}$, yielding $\left|\alpha^{i,ii}_{1\lim}\right| = \bar{\hat{\beta}}^{i,ii}_{(2)\lim} = \left|\alpha^{i,ii}_{2\lim}\right| = \bar{\hat{\beta}}^{i,ii}_{(1)\lim}$. The parameters with superscripts "i", "ii" are relevant to $\hat{k}^{i}_{\lim}(\bar{r}_0)$ and $\hat{k}^{ii}_{\lim}(\bar{r}_0)$, resp.

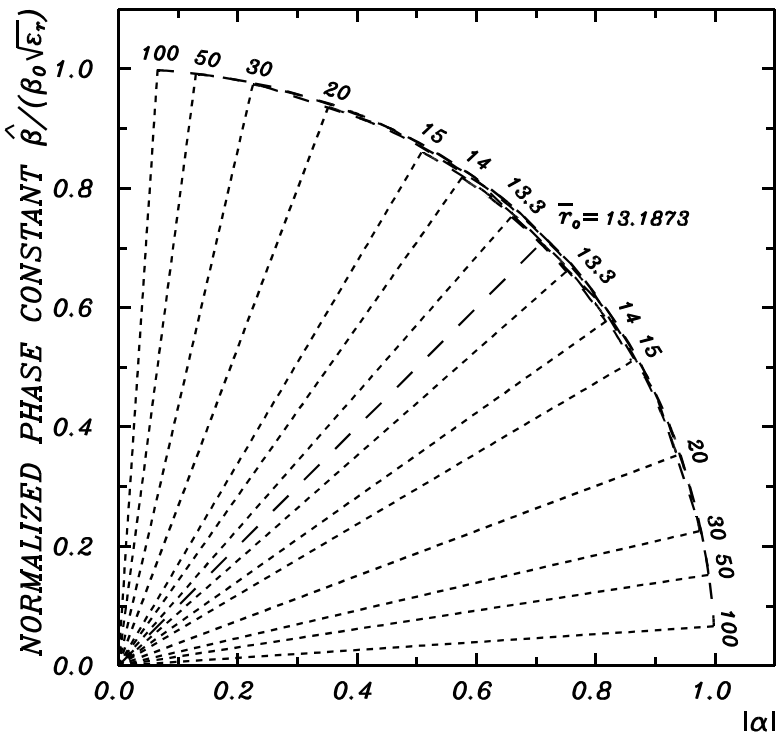

Figure 2. $\bar{\hat{\beta}}(|\alpha|)$ curves of slow TE_{01} mode for $0<|\alpha|<1$.

In case $|\alpha|>3$ the vertical (horizontal) straight line with equation $|\alpha|=3$ ($\bar{\hat{\beta}}=3$) restricts from the left (below) the area in which the wave could be sustained, as seen from Fig. 1. Rightwards and upwards this area is unlimited. The $\bar{\hat{\beta}}(|\alpha|)$ characteristics are infinite in both directions, concave with respect to the bisectrix curves. Propagation here is observed for any $\bar{r}_0$ from 0 to $+\infty$. If the normalized guide radius diminishes (increases), the phase characteristics move away from (towards) the asymptotes. It is interesting to note that even for relatively small $\bar{r}_0$ (e.g. $\bar{r}_0 \geq 1$) the $\bar{\hat{\beta}}(|\alpha|)$ curves almost coincide with the lines $|\alpha|=3$, $\bar{\hat{\beta}}=3$. The constant $\bar{\hat{\beta}}$ may become very large if $\bar{r}_0$ gets small or/and $|\alpha|\to 3$. Thus, under these conditions the wave may be slowed down as much as it is prescribed.

Provided $0<|\alpha|<1$ the phase characteristic, answering to $\bar{r}_0 = 2\hat{L}(\hat{c},n)$ degenerates into a point which lies at the bisectrix (cf. Fig. 2). For any $\bar{r}_0$, larger than this minimum value, the $\bar{\hat{\beta}}(|\alpha|)$ curves are finite arcs, convex regarding it. They lengthen when $\bar{r}_0$ increases. Their ends are connected with the origin by dotted lines.

The angle formed grows with $\bar{r}_0$ and for $\bar{r}_0 \to +\infty$ tends to $\pi/2$. The phase arcs lie slightly above the unit circle which determines the normalized natural propagation constant $\bar{\beta}_\infty = \sqrt{1-\alpha^2}$ in the unbounded azimuthally magnetized ferrite medium [8]. Consequently, the guide provides now a small slowing down of the wave, since $\bar{\hat{\beta}}$ is inconsiderably larger than $\bar{\beta}_\infty$.

5. DISCUSSIONS AND CONCLUSIONS

A property of the real zeros of Kummer CHFs is found out. Combined with known facts for the functions and their zeros, due to Tricomi [20], [21], the propagation conditions and the main features of the slow TE_{0n} modes in a circular ferrite waveguide with azimuthal magnetization are determined: possibility for transmission in case of negative sign of the latter only in two areas, corresponding to weak and strong anisotropy. It should be noted that such waves cannot exist, if the filling degenerates into isotropic dielectric. The characteristics established exhibit the inherent nonreciprocity of the structure in contrast to the unbounded azimuthally magnetized ferrite medium which is reciprocal. As known, the latter sustains waves solely provided $0 < |\alpha| < 1$ [1]. The potentiality for TE_{0n} modes propagation also for values of $|\alpha|$ larger than the limit 1 (larger than 3 in case of TE_{01} mode) is an important peculiarity of the waveguide, due to the influence of its wall. The comparison of the behaviour of the slow guided waves with that of the normal ones reveals a substantial difference. The normal TE_{0n} modes are supported only in one zone, subject to the condition $0 < |\alpha| < 1$ but for both directions of magnetization. Switching it produces differential phase shift for them in discrete amounts [7], [8], [11] whereas in the case investigated here, it results in an effect of transmitting or ceasing the wave. For this reason the structure considered under slow waves excitation is appropriate for current controlled switch or isolator solely but not also for phase shifter as when normal propagation takes place along it.

REFERENCES

[1] H. Suhl and L.R. Walker, “Topics in guided wave propagation through gyromagnetic media. Part II: Transverse magnetization and the non-reciprocal helix”, *Bell System Tech. J.*, vol. 33, no. 4, pp. 939-986, July 1954.

[2] N. Kumagai and K. Takeuchi, “Circular electric waves propagationg through the circular waveguide containing a circumferentially magnetized ferrite cylinder”, *IRE Wescon Convention Record*, pt. I, pp. 123-130, 1958.

[3] M.E. Averbuch, “Difference functions for evaluation of layered azimuthally magnetized ferrite waveguides”, *in Conf. Proc. V Intnl. Conf. on Microwave Ferrites*, Vilnius, USSR, Oct. 1980, vol. 4, pp. 126-133, (in Russian).

[4] T. Obunai and K. Hakamada, “Slow surface wave propagation in an azimuthally-magnetized millimeter-wave solid-plasma coaxial waveguide”, *Jap. J. Appl. Phys.*, vol. 23, no. 8, pp. 1032-1037, Aug. 1984.

[5] K.P. Ivanov and G.N. Georgiev, "On a class of electromagnetic wave functions for propagation along the circular gyrotropic waveguide", *IEEE Trans. Microwave Theory Tech.*, vol. MTT-34, no. 8, pp. 853-862, Aug. 1986.

[6] K.P. Ivanov, G.N. Georgiev, "Kummer and Tricomi functions for analysis of circular gyrotropic waveguides with azimuthal magnetisation", *Proc. Inst. Elec. Eng.*, vol. 135, pt. H (Microwaves, Antennas and Propagation), no. 2, pp. 125-128, Apr. 1988.

[7] J. Helszajn, *Ferrite Phase Shifters and Control Devices*, McGraw-Hill, London: K.P. Ivanov, G.N. Georgiev, “Azimuthally magnetized circular ferrite waveguides”, ch. 14, pp. 262-288, 1989.

[8] K.P. Ivanov and G.N. Georgiev, “Some properties of the circular waveguide with azimuthally magnetized ferrite”, *J. Appl. Phys.*, vol. 67, no. 10, pp. 6529-6537, May 1990.

[9] A.A.P. Gibson, R. Sloan, L.E. Davis, D.K. Paul, “Double valued phase constants in gyrotropic waveguides”, *Proc. Inst. Elec. Eng.*, pt. H, (Microwaves, Antennas and Propagation); vol. 138, pp. 258-260, June 1991.

[10] L.E. Davis, R. Sloan, and D.K. Paul, “TM_{0m} modes in transversely magnetised semiconductor-filled coaxial waveguide and parallel plates”, *Proc. Inst. Elec.*

Eng., pt. H (Microwaves, Antennas and Propagation), vol. 140, no. 3, pp. 211-218, June 1993.

[11] G.N. Georgiev and M.N. Georgieva-Grosse, "Formulae for differential phase shift computation in an azimuthally magnetized circular ferrite waveguide", *in Conf. Proc. Millennium Conf. on Antennas and Propagation AP-2000*, Davos, Switzerland, 9-14 Apr. 2000, paper 1002.

[12] D.M. Bolle and G.S. Heller, "Theoretical considerations on the use of circularly symmetric TE modes for digital ferrite phase shifters", *IEEE Trans. Microwave Theory Tech.*, vol. MTT-13, no. 4, pp. 421-426, July 1965. See also D.M. Bolle and N. Mohsenian, "Correction", *IEEE Trans. Microwave Theory Tech.*, vol. MTT-34, no. 4, p. 427, Apr. 1986.

[13] W.J. Ince and G.N. Tsandoulas, 'Modal inversion in circular waveguides. Part II: Application to latching nonreciprocal phasers", *IEEE Trans. Microwave Theory Tech.*, vol. MTT-19, no. 4, pp. 393-400, Apr. 1971.

[14] F.J. Bernues and D.M. Bolle, "A study of modes in circular waveguide with azimuthally magnetized ferrite", Division of Engineering, Brown University, Providence, R.I., NSF Research Grant NSF-GK 2351/1, Washington, D.C., Apr. 1971.

[15] F.J. Bernues and D.M. Bolle, "The digital twin-ferrite-toroid circular waveguide phaser", *IEEE Trans. Microwave Theory Tech.*, vol. MTT-21, no. 12, pp. 842-845, Dec. 1973.

[16] R.S. Mueller and F.J. Rosenbaum, "Propagation along azimuthally magnetized ferrite-loaded circular waveguides", *J. Appl. Phys.*, vol. 48, no. 6, pp.2601-2603, June 1977.

[17] S.N. Samaddar, "Special functions associated with azimuthally magnetized ferrite rod phase shifters", *J.Appl. Phys.*, vol. 50, no. 1, pp.518-520, Jan. 1979.

[18] L.J. Slater, *Confluent Hypergeometric Functions*, Cambridge, England: Cambridge Univ. Press, 1960, ch. 1, ch. 2.

[19] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Cambridge, England: Cambridge Univ. Press, 1965, ch. XVI.

[20] F.G. Tricomi, *Funcioni Ipergeometriche Confluenti*, Edizioni Cremonese, Rome, Italy, 1954.

[21] F.G. Tricomi, *Fonctions Hypergéométrique Confluentes*, Gauthier-Villars, Paris, France, 1960.