THE CALCULATION OF LEVITATION FORCE BETWEEN AXIALLY POLARIZED MAGNETIC DISKS BY CALCULATING FUNCTION

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Abstract

In this article, the analytic expression of levitation force between axially polarized magnetic disks is given. This is achieved through calculating the function of levitation force between two uniformly charged coaxial ring-shaped disks. The calculating function can be studied or tabulated. By using approximate expressions, a rapid calculation can be performed. The presented expression is tested with the data from literature.

1. INTRODUCTION

The study of the force between two permanent magnets is important both theoretically and practically, as many applications need solutions of problems related to it. The reason is that permanent magnets can be used in different applications such as electrical machines, magnetic joints for mechanical transmission, magnetic suspensions, etc. Moreover, magnetic joints with permanent magnets are the only devices used for the transmission of forces and torque between two aggregates installed in different environments (liquid–fluid, high pressure–vacuum) divided with non-ferromagnetic walls [1].

There are numerous techniques for analyzing permanent magnet devices. Most of them are based on the finite elements method [2-4]. The market offers software based on this method, which can calculate a large number of parameters needed to analyze the performance of permanent magnets. The method of finite elements, however, expresses serious difficulties when applied for open zones, despite continuous efforts for its modification and improvement. Therefore, it is better to avoid it in some particular cases, and to use other options instead. A simple formula for the calculation of levitation force between two coaxial uniformly polarized magnetic disks is proposed in [5]. It simplifies the calculation compared to the method of finite elements. This formula has practical importance and represents a special case for the numerical calculation of the integral of the above-mentioned force.

The scope of this article is to propose an analytic expression for the calculation of the levitation force between two coaxial uniformly polarized magnetic disks. In our case, we will base on the concept of the calculating function given in [6] as a normalized primitive of the integral of needed force.

2. MATHEMATICAL MODEL

The analysis to find out the analytic expression of the levitation force between two coaxial uniformly polarized magnetic disks is based on the assumption of ideal magnets that are characterized by fixed and uniform polarization. Experiments confirm that this assumption is very realistic in the case of modern high strength materials such as NdFeB are used [7, 8].

There are two mathematical models to approach permanent magnets: the model of electrical currents and the model of static magnetic charges. In our analysis we will focus on the second one, which models the permanent magnets as volume (ρ_m) and surface (σ_m) distribution of magnetic charges. The volume and surface distribution of magnetic charges can be calculated using the following formulas:

$$\rho_m(\mathbf{x}) = -\nabla\mathbf{M}(\mathbf{x}) \tag{1a}$$

$$\sigma_m(\mathbf{x}) = \mathbf{M}(\mathbf{x})\cdot\mathbf{n} \tag{1b}$$

In the above formulas, **M** is the vector of magnetization at the point with coordinate **x,** and **n** denotes the unit vector, perpendicular with the surface of the magnet.

For permanent magnets with uniform magnetization, based on (1a), the distribution of volume magnetic charges is zero

$$\rho_m \equiv 0 \tag{2}$$

and the model of permanent magnets can be expressed only with the distribution of their surface (σ_m) magnetic charges. The magnetic field (**B**) of these permanent magnets, based on the analogy with electrostatic field, can be computed using the following integral expression:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_S \frac{\sigma_m(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\,dS' \tag{3}$$

where S defines the surface that bounds the magnet.

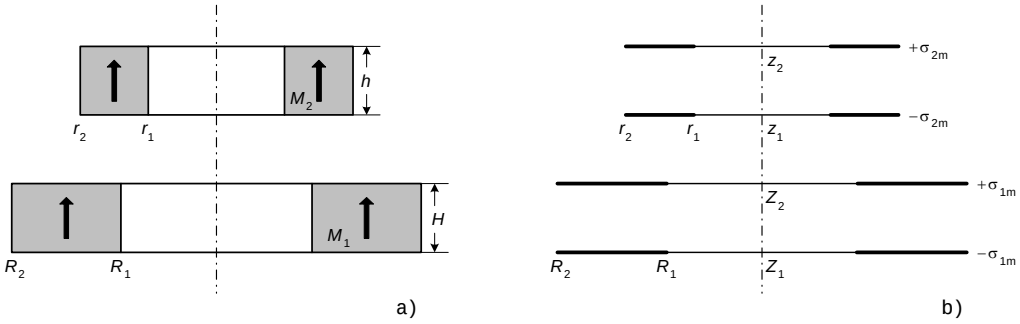

Figure 1. The geometry of magnets (a) and their mathematical model (b).

The force (**F**) between two magnets can be computed using the following integral expression:

$$\mathbf{F} = \int_S \sigma_m(\mathbf{x})\mathbf{B}_{\text{ex}}(\mathbf{x})\mathbf{dS} \tag{4}$$

where $\mathbf{B}_{\text{ex}}$ is the external magnetic field due to other currents and/or magnets.

Figure 1a presents the geometry and the reciprocal position of two coaxial ring-shaped and uniformly axially polarized magnets. Figure 1b presents a mathematical model, derived from the above model, of four coaxial ring-shaped and uniformly charged disks, two for each magnet, and with alternate surface density. These disks are equal with the front of the magnets and their surface density can be computed using the following expression:

$$\sigma_{1m} = \mathbf{M}_1\mathbf{k} \qquad \sigma_{2m} = \mathbf{M}_2\mathbf{k} \tag{5}$$

where $\mathbf{M_1}$ and $\mathbf{M_2}$ denote the vector of magnetization for each magnet, and **k** is the unit vector according to the axis of the magnets. So, the computation of the force between two magnets can be reduced in the computation of the sum of forces between charged disks of each magnet. In our case, because of symmetry of the problem, only axial component of force exists. If we define as f_{mn} the force between two charged disks, then the axial force between two magnets can be computed as:

$$F = \sum_{m=1}^{2}\sum_{n=1}^{2}(-1)^{n+m} f_{mn} \tag{6}$$

where index m stands for the first magnet and index n stands for the second one.

3. THE FORCE BETWEEN TWO MAGNETS

The calculation of the force between two magnets is based on the calculating function of the axial force between two coaxial uniformly charged disks. Figure 2 considers two disks, one from each magnet of figure 1. The axial force between these disks can be derived from (4), by substituting $\mathbf{B}_{ex}$ with its expression given in (3).

$$f_{mn} = \frac{\mu_0 \sigma_{1m} \sigma_{2m}}{4\pi} \int_{R_1}^{R_2} dR \int_{r_1}^{r_2} dt \int_0^{2\pi} \frac{2\pi R t D_{mn} d\theta}{\left(R^2 + t^2 + D_{mn}^2 - 2Rt\cos\theta\right)^{3/2}} \tag{7}$$

where:

$$D_{mn} = z_n - Z_m \tag{8}$$

is the distance between two disks plans.

The calculating function of the axial force between two coaxial disks can be defined as follows:

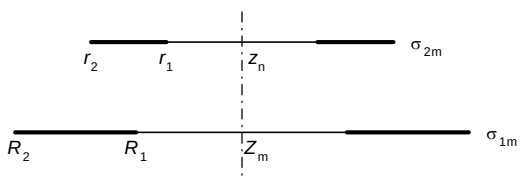

Figure 2. Two coaxial uniformly charged disks

$$\mathbf{F}(r,z) = \int_0^1 dR \int_0^r dt \int_0^{2\pi} \frac{2\pi R t z d\theta}{\left(R^2 + t^2 + z^2 - 2Rt\cos\theta\right)^{3/2}} \tag{9}$$

The force (7) can be expressed by using its calculating function (9) through the following relationship:

$$f_{mn} = \frac{\mu_0 \sigma_{1m} \sigma_{2m}}{4\pi} \sum_{i=1}^{2} \sum_{j=1}^{2} (-1)^{i+j} R_i^2 \mathbf{F}\left(\frac{r_j}{R_i}, \frac{D_{mn}}{R_i}\right) \tag{10}$$

The substitution of (10) in (6) defines the quantity of the axial force between two magnets expressed with the calculating function $\mathbf{F}(r, z)$:

$$F = \frac{\mu_0 M_1 M_2}{4\pi} \sum_{m=1}^{2}\sum_{n=1}^{2}\sum_{i=1}^{2}\sum_{j=1}^{2} (-1)^{m+n+i+j} R_i^2 \mathbf{F}\left(\frac{r_j}{R_i}, \frac{z_n - Z_m}{R_i}\right) = \frac{\mu_0 M_1 M_2}{4\pi} \sum_{\nu=1}^{16} A_\nu \mathbf{F}(r_\nu, z_\nu) \quad (11)$$

where $\nu = 8n + 4m + 2i + j - 14$, $A_\nu = (-1)^{m+n+i+j} R_i^2$, $r_\nu = r_j/R_i$ and $z_\nu = (z_n - Z_m)/R_i$.

4. THE STUDY OF CALCULATING FUNCTION F(*r*, *z*)

The study of calculating function **F**(*r*, *z*) includes the evaluation of its analytic expression, the behavior of the function in special points, and the tabulation of its values for a quick computation. The appendix of the article displays the table of values of the calculating function **F**(*r*, *z*). For ranges of values that are not included in the table some approximate expressions are given.

4.1. Analytical Expression

The analytical expression of the calculating function can be obtained by integration of (9) firstly by variable *t*, then by variable *R*, and at last by variable θ. The last integration can be performed only by using the Legendre's elliptic integrals. The final analytic expression of the calculating function after calculations based on [9, 10] is:

$$\begin{aligned}\mathbf{F}(r,z) = 2\pi^2 \operatorname{sgn} z(1+r^2) - 2\pi z \Bigg\{ & \frac{\sqrt{r}}{k}\left[(1+k'^2)\mathbf{K}(k) - 2\mathbf{E}(k)\right] + \\ & + \frac{k}{4}\sqrt{r}\left[h_1\left(\sqrt{r^2+z^2}+1\right)\mathbf{\Pi}(h_1,k) - h_2\left(\sqrt{r^2+z^2}-1\right)\mathbf{\Pi}(h_2,k)\right] + \\ & + \frac{k}{4}\sqrt{r}\left[h_3\left(\sqrt{1+z^2}+r\right)\mathbf{\Pi}(h_3,k) - h_4\left(\sqrt{1+z^2}-r\right)\mathbf{\Pi}(h_4,k)\right]\end{aligned} \quad (12)$$

where **K**(*k*) and **E**(*k*) are the complete elliptic integrals of first and second kind respectively:

$$\mathbf{K}(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}} \qquad \mathbf{E}(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2 x}\,dx$$

with module *k* equal to:

$$k^2 = \frac{4r}{(1+r)^2 + z^2} \quad (13)$$

and **Π**(*h*, *k*) is the complete elliptic integral of third kind

$$\mathbf{\Pi}(h,k) = \int_0^{\pi/2} \frac{dx}{\left(1+h\sin^2 x\right)\sqrt{1-k^2\sin^2 x}}$$

with module h as the follow:

$$h_1 = \frac{2r}{\sqrt{r^2 + z^2} - r}, \quad h_2 = -\frac{2r}{\sqrt{r^2 + z^2} + r}, \quad h_3 = \frac{2}{\sqrt{1 + z^2} - 1}, \quad h_4 = \frac{2}{\sqrt{1 + z^2} + 1} \tag{14}$$

4.2. The behavior of the function **F**(r, z) for z values near zero

The calculating function **F**(r, z) has a jump in the point $z = 0$. To study its behavior in the points $z = 0\pm$, we have to study the behavior of the elliptic integral of the third kind $\mathbf{\Pi}$ in special points [11]. So, the behavior of the function $\mathbf{\Pi}$ for $h \to \propto$ is:

$$\frac{k^2\mathbf{K}(k)}{k^2 + h} + \frac{\pi}{2}\frac{\sqrt{h}}{\sqrt{(1+h)(k^2+h)}} \tag{15}$$

and for $h \to -1$ ($|h| < 1$) is:

$$-\frac{\pi}{2}\frac{h}{\sqrt{(1+h)(-h-k^2)}} \tag{16}$$

Based on (15) we can write:

$$\lim_{z\to 0} h_1\mathbf{\Pi}(h_1, k) = k^2\mathbf{K}(k) + \frac{\pi r}{|z|}, \qquad \lim_{z\to 0} h_3\mathbf{\Pi}(h_3, k) = k^2\mathbf{K}(k) + \frac{\pi}{|z|} \tag{17}$$

and based on (16):

$$\lim_{z\to 0} h_2\mathbf{\Pi}(h_2, k) = -\frac{\pi r}{|z|k'}, \qquad \lim_{z\to 0} h_4\mathbf{\Pi}(h_4, k) = -\frac{\pi}{|z|k'} \tag{18}$$

The values of the function **F**(r, z) for z near zero can be obtained by substituting the above expressions in (12):

$$\mathbf{F}(r, 0\pm) = \pm 2\pi^2 \begin{cases} r^2 & \text{për} \quad r \le 1 \\ 1 & \text{për} \quad r \ge 1 \end{cases} \tag{19}$$

4.3. Approximate Expressions

The tabulate calculation of the function **F** is limited, because it does not include all the values of its variables. To eliminate this handicap, some approximate expressions of **F** are presented below. These expressions can be used only for $r \le 1$. The values of **F** for $r > 1$ can be calculated by:

$$\mathbf{F}\left(\frac{1}{r}, z\right) = \frac{1}{r^2}\mathbf{F}(r, rz) \tag{20}$$

The verification of the above formula derives directly from (8).

– for small values of r ($r \to 0$) the following approximate expression can be used:

$$\mathbf{F}(r,z) \cong \pi^2 r^2 \operatorname{sgn} z \left(A_r - B_r r^2 \right) \tag{21}$$

where A_r and B_r are as follow:

$$A_r = 2\left(1 - \frac{|z|}{\sqrt{1+z^2}}\right) \qquad B_r = \frac{3|z|}{4\left(1+z^2\right)^{5/2}} \tag{22}$$

The use of expression (21) to compute the value of **F** gives an error that does not exceed 0.05% for $r \le 0.3$, and decreases to 0.01% for $r \le 0.2$. For $r \le 0.1$, the approximate value of **F** computed with (21) has six numbers equal to the correct one.

– for high values of z ($z \to \propto$) the following approximate expression can be used:

$$\mathbf{F}(r,z) \cong \frac{\pi^2 r^2}{z^2} \operatorname{sgn} z \left(1 - \frac{A_z}{z^2} + \frac{B_z}{z^4} - \frac{C_z}{z^6} + \frac{D_z}{z^8}\right) \tag{23}$$

where A_z, B_z, C_z and D_z can be computed as:

$$A_z = \tfrac{3}{4}\left(1+r^2\right) \qquad B_z = \tfrac{5}{8}\left(1+3r^2+r^4\right) \qquad C_z = \tfrac{35}{64}\left(1+6r^2+6r^4+r^6\right)$$

$$D_z = \tfrac{63}{128}\left(1+10r^2+20r^4+10r^6+r^8\right) \tag{24}$$

The use of expression (23) to compute the value of **F** gives an error that does not exceed 0.1% for $z \ge 3$. For $z \ge 5$, the approximate value of **F** computed with (23) has six numbers equal to the correct one and for $z \ge 10$ it has until eight numbers equal.

5. NUMERICAL EXAMPLE

The analytical expression (11) presented above was tested against numerical data that have been taken from [5], in which a numerical formula for the calculation of the levitation force between two coaxial uniformly polarized magnets is presented. The numerical input values are: M = 430000 A/m, $R_1 = r_1$ = 20 mm, $R_2 = r_2$ = 40 mm, $H = h$ = 10 mm. The magnets are arranged in a way that their magnetization is in opposite direction and the distance from their nearest fronts is assumed 10 mm. The value of M corresponds to bonded NdFeB material.

Based on (11) the following table is filled.

ν	A_ν	r_ν	z_ν	$\mathbf{F}(r_\nu, z_\nu)$
1	+0.0004	1	1	4.583657
2	–0.0004	2	1	10.346380
3	–0.0016	0.5	0.5	2.586595

4	+0.0016	1	0.5	8.331688
5	–0.0004	1	0.5	8.331688
6	+0.0004	2	0.5	14.496096
7	+0.0016	0.5	0.25	3.624024
8	–0.0016	1	0.25	11.950434
9	–0.0004	1	1.5	2.799282
10	+0.0004	2	1.5	7.435360
11	+0.0016	0.5	0.75	1.858840
12	–0.0016	1	0.75	6.082812
13	+0.0004	1	1	4.583657
14	–0.0004	2	1	10.346380
15	–0.0016	0.5	0.5	2.586595
16	+0.0016	1	0.5	8.331688

In this table the values of **F** for $r > 1$ are computed based on (20).

Substituting the above values in (11), the value of axial force derived is:

$F = 36.726623$ [N]

The formula presented in [5] by Furlani gives for the same inputs the value of levitation force $F = 36.766451$, which is very near to the above value computed with the analytical expression presented in this article.

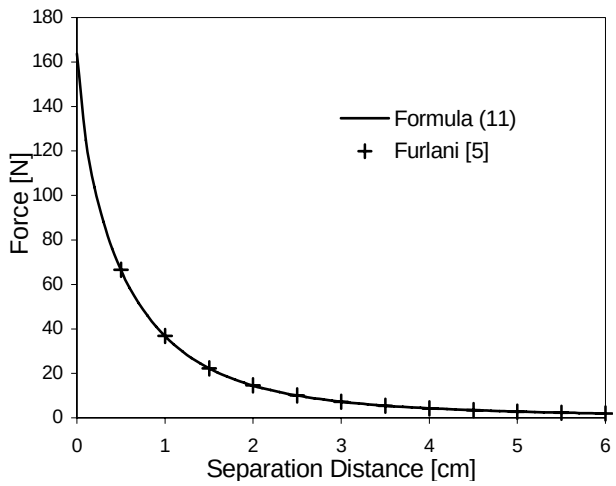

Figure 3. The axial force versus separation distance between two magnets.

Figure 3 presents the levitation force between two above magnets versus their separation distance. Crosses present the values computed using the formulae of [5]. The comparison of data shows a good accordance.

6. CONCLUSIONS

An analytical expression has been derived for calculating the levitation force between two coaxial uniformly polarized permanent magnets, which, within the assumptions accepted, increases the accuracy and speed of computation. Knowing the computation procedures of elliptic integrals allows a quick programming of the analytical expression.

The analytical expression (11) can be used in a wide range of computations. Comparing it with the formula of Furlani [5] the following priorities can be distinguished: (a) it includes the case where two magnets are in contact with each other ($h = 0$), and (b) it includes every kind of magnet instead of [5], which includes only similar magnets.

The tabulation of calculating function offers a good possibility for a quick and low cost computation.

REFERENCES

[1] Parker R.J., *Advanced in Permanent Magnetism*, New York, Willey, 1990

[2] Silvester P.P., Ferrari R.L., *Finite Elements for Electrical Engineers*, Cambridge University Press, New York, 1986

[3] Lowther D.A., Silvester P.P., *Computer Aided Design in Magnetics*, Springer Verlag, New York, 1986

[4] Humphries S.Jr., *Field Solutions on Computers*, CRC Press, New York, 1998

[5] Furlani E.P., *A Formula for the Levitation Force between Magnetic Disks*, IEEE Trans. on Magnetics, Vol. 29, No 6, 4165-4169, 1993

[6] Braneshi M., *The calculating functions of magnetic field of circular coils and the method calculation of magnetic field by them*, B.SH.T., Nr. 1, 47-55, 1996

[7] Furlani E.P., *Formulas for the Force and Torque of Axial Couplings*, IEEE Trans. on Magnetics, Vol. 29, No 5, 2295-2301, 1993

[8] Yao Y.D., Chion G.J., Huang D.R., Wang S.J., *Theoretical Computations for the Torque of Magnetic Coupling*, IEEE Trans. on Magnetics, Vol. 31, No 3, 1881-1884, 1995

[9] Dwight H.B., *Tables of Integrals and Other Mathematical Data*, The Macmillan Company, New York, 1961.

[10] Gradshteyn I.S., Ryzhik I.M., *Table of Integrals, Series and Products*, New York, Academic Press, 1990.

[11] Selfridge R.G., Maxfield J.E., *A table of the incomplete elliptic integral of the third kind*, Dover publication INC, New York, 1958.

APPENDIX

Table of values of calculating function F(r, z).
Case $0 < r \le 1$ and $0 \le z \le 1$.

z	r									
	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.00	0.049348	0.197392	0.444132	0.789568	1.233701	1.776529	2.418053	3.158273	3.997190	4.934802
0.05	0.046881	0.187498	0.421764	0.749536	1.170597	1.684656	2.291322	2.990084	3.780276	4.661022
0.10	0.044433	0.177678	0.399569	0.709819	1.108011	1.593570	2.165735	2.823516	3.565621	4.390366
0.15	0.042021	0.168005	0.377711	0.670723	1.046435	1.504019	2.042380	2.660099	3.355343	4.125737
0.20	0.039662	0.158545	0.356343	0.632526	0.986323	1.416685	1.922236	2.501199	3.151300	3.869630
0.25	0.037369	0.149358	0.335600	0.595473	0.928068	1.332158	1.806138	2.347957	2.955013	3.624024
0.30	0.035157	0.140492	0.315596	0.559769	0.871998	1.250916	1.694753	2.201263	2.767631	3.390350
0.35	0.033034	0.131989	0.296421	0.525577	0.818365	1.173325	1.588576	2.061755	2.589931	3.169507
0.40	0.031008	0.123877	0.278143	0.493013	0.767350	1.099636	1.487934	1.929828	2.422358	2.961937
0.45	0.029084	0.116179	0.260805	0.462154	0.719064	1.029998	1.393003	1.805668	2.265069	2.767707
0.50	0.027266	0.108903	0.244431	0.433037	0.673559	0.964466	1.303829	1.689284	2.117991	2.586595
0.55	0.025553	0.102054	0.229027	0.405667	0.630831	0.903019	1.220352	1.580544	1.980877	2.418170
0.60	0.023946	0.095628	0.214582	0.380021	0.590836	0.845575	1.142428	1.479213	1.853352	2.261861
0.65	0.022441	0.089616	0.201073	0.356056	0.553496	0.792003	1.069853	1.384979	1.734959	2.117009
0.70	0.021037	0.084004	0.188469	0.333710	0.518707	0.742140	1.002380	1.297482	1.625186	1.982909
0.75	0.019728	0.078775	0.176731	0.312909	0.486347	0.695798	0.939733	1.216333	1.523497	1.858840
0.80	0.018510	0.073910	0.165814	0.293575	0.456285	0.652778	0.881623	1.141129	1.429349	1.744087
0.85	0.017378	0.069390	0.155673	0.275620	0.428383	0.612872	0.827755	1.071466	1.342207	1.637956
0.90	0.016326	0.065193	0.146259	0.258958	0.402501	0.575872	0.777839	1.006950	1.261550	1.539784
0.95	0.015351	0.061298	0.137524	0.243503	0.378501	0.541577	0.731588	0.947199	1.186885	1.448944
1.00	0.014446	0.057684	0.129422	0.229170	0.356248	0.509787	0.688732	0.891851	1.117743	1.364849

z	r									
	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
0.00	5.971111	7.106115	8.339816	9.672212	11.103305	12.633094	14.261578	15.988759	17.814636	19.739209
0.05	5.631156	6.689097	7.832635	9.058571	10.362032	11.735080	13.163506	14.618145	16.026061	17.178203
0.10	5.295527	6.278113	7.334013	8.457396	9.639659	10.867492	12.119068	13.356454	14.512402	15.485788
0.15	4.968184	5.878589	6.851442	7.879161	8.951059	10.051662	11.158105	12.236660	13.240674	14.117991
0.20	4.652447	5.494902	6.390640	7.331225	8.305332	9.297652	10.287624	11.248581	12.148978	12.958173
0.25	4.350838	5.130186	5.955356	6.817759	7.706397	8.607298	9.503100	10.373278	11.195755	11.950434
0.30	4.065063	4.786365	5.547553	6.340341	7.154566	7.977963	8.796184	9.593322	10.353211	11.061530
0.35	3.796087	4.464334	5.167797	5.898745	6.648035	7.405084	8.158058	8.894395	9.601752	10.269282
0.40	3.544257	4.164186	4.815663	5.491619	6.183941	6.883543	7.580596	8.264968	8.926894	9.557785
0.45	3.309445	3.885439	4.490085	5.116990	5.759002	6.408306	7.056653	7.695707	8.317516	8.915046
0.50	3.091179	3.627232	4.189620	4.772603	5.369877	5.974694	6.580052	7.178968	7.764816	8.331688
0.55	2.888757	3.388471	3.912640	4.456120	5.013352	5.578471	6.145471	6.708411	7.261665	7.800174
0.60	2.701333	3.167941	3.657448	4.165248	4.686425	5.215850	5.748318	6.278710	6.802176	7.314318
0.65	2.527983	2.964379	3.422363	3.897804	4.386341	4.883462	5.384617	5.885344	6.381411	6.868952
0.70	2.367755	2.776530	3.205762	3.651746	4.110595	4.578319	5.050908	5.524438	5.995180	6.459699
0.75	2.219703	2.603176	3.006115	3.425188	3.856923	4.297768	4.744169	5.192646	5.639884	6.082812
0.80	2.082907	2.443156	2.821989	3.216403	3.623283	4.039454	4.461745	4.887053	5.312410	5.735050
0.85	1.956487	2.295382	2.652062	3.023814	3.407837	3.801280	4.201299	4.605109	5.010041	5.413590
0.90	1.839614	2.158838	2.495114	2.845989	3.208932	3.581375	3.960758	4.344569	4.730393	5.115953
0.95	1.731513	2.032585	2.350031	2.681626	3.025080	3.378072	3.738285	4.103447	4.471362	4.839955
1.00	1.631464	1.915758	2.215793	2.529547	2.854943	3.189875	3.532242	3.879978	4.231084	4.583657

Table of values of calculating function F(r, z).
Case $0 < r \le 1$ and $z \ge 1$.

$\frac{1}{z}$	r									
	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.05	0.000062	0.000246	0.000554	0.000985	0.001539	0.002216	0.003016	0.003939	0.004985	0.006154
0.10	0.000245	0.000980	0.002204	0.003917	0.006120	0.008811	0.011990	0.015655	0.019808	0.024445
0.15	0.000546	0.002184	0.004912	0.008730	0.013635	0.019626	0.026699	0.034852	0.044079	0.054376
0.20	0.000958	0.003832	0.008619	0.015316	0.023916	0.034412	0.046796	0.061058	0.077185	0.095164
0.25	0.001473	0.005891	0.013248	0.023535	0.036738	0.052842	0.071826	0.093666	0.118336	0.145805
0.30	0.002081	0.008320	0.018706	0.033222	0.051842	0.074533	0.101258	0.131970	0.166618	0.205142
0.35	0.002770	0.011074	0.024892	0.044195	0.068938	0.099068	0.134516	0.175205	0.221047	0.271941
0.40	0.003529	0.014105	0.031700	0.056264	0.087730	0.126012	0.171006	0.222589	0.280624	0.344956
0.45	0.004345	0.017367	0.039022	0.069239	0.107919	0.154937	0.210140	0.273352	0.344372	0.422976
0.50	0.005208	0.020813	0.046755	0.082935	0.129218	0.185429	0.251359	0.326764	0.411370	0.504869
0.55	0.006106	0.024399	0.054800	0.097179	0.151356	0.217101	0.294140	0.382151	0.480772	0.589603
0.60	0.007030	0.028085	0.063067	0.111811	0.174087	0.249604	0.338012	0.438905	0.551823	0.676261
0.65	0.007969	0.031835	0.071476	0.126688	0.197189	0.282622	0.382555	0.496489	0.623859	0.764042
0.70	0.008916	0.035617	0.079954	0.141684	0.220471	0.315883	0.427406	0.554439	0.696307	0.852262
0.75	0.009865	0.039402	0.088439	0.156691	0.243763	0.349152	0.472252	0.612360	0.768684	0.940345
0.80	0.010808	0.043167	0.096879	0.171616	0.266926	0.382230	0.516831	0.669921	0.840586	1.027814
0.85	0.011742	0.046892	0.105229	0.186383	0.289842	0.414953	0.560927	0.726849	0.911682	1.114279
0.90	0.012661	0.050562	0.113455	0.200930	0.312416	0.447187	0.604364	0.782923	0.981706	1.199428
0.95	0.013563	0.054162	0.121526	0.215205	0.334571	0.478826	0.647002	0.837969	1.050447	1.283014
1.00	0.014446	0.057684	0.129422	0.229170	0.356248	0.509787	0.688732	0.891851	1.117743	1.364849

$\frac{1}{z}$	r									
	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
0.05	0.007446	0.008860	0.010397	0.012057	0.013839	0.015743	0.017770	0.019918	0.022189	0.024582
0.10	0.029568	0.035173	0.041260	0.047829	0.054876	0.062401	0.070403	0.078879	0.087827	0.097246
0.15	0.065740	0.078163	0.091641	0.106166	0.121733	0.138332	0.155958	0.174601	0.194253	0.214904
0.20	0.114981	0.136619	0.160061	0.185287	0.212277	0.241011	0.271464	0.303613	0.337432	0.372896
0.25	0.176039	0.209001	0.244651	0.282948	0.323845	0.367293	0.413244	0.461642	0.512434	0.565563
0.30	0.247478	0.293557	0.343302	0.396633	0.453465	0.513706	0.577263	0.644038	0.713930	0.786834
0.35	0.327780	0.388444	0.453807	0.523733	0.598081	0.676700	0.759436	0.846129	0.936614	1.030723
0.40	0.415418	0.491825	0.573985	0.661689	0.754722	0.852857	0.955862	1.063497	1.175515	1.291670
0.45	0.508923	0.601948	0.701772	0.808101	0.920627	1.039029	1.162981	1.292145	1.426181	1.564747
0.50	0.606930	0.717195	0.835282	0.960794	1.093314	1.232413	1.377653	1.528587	1.684765	1.845736
0.55	0.708208	0.836118	0.972838	1.117851	1.270618	1.430586	1.597192	1.769865	1.948035	2.131132
0.60	0.811665	0.957447	1.112983	1.277622	1.450690	1.631498	1.819345	2.013527	2.213340	2.418088
0.65	0.916361	1.080094	1.254476	1.438714	1.631985	1.833451	2.042266	2.257580	2.478548	2.704339
0.70	1.021494	1.203137	1.396279	1.599969	1.813229	2.035063	2.264465	2.500433	2.741975	2.988119
0.75	1.126394	1.325815	1.537535	1.760443	1.993394	2.235223	2.484760	2.740841	3.002319	3.268077
0.80	1.230510	1.447501	1.677551	1.919377	2.171659	2.433057	2.702229	2.977842	3.258588	3.543201
0.85	1.333394	1.567694	1.815772	2.076168	2.347381	2.627889	2.916169	3.210713	3.510049	3.812755
0.90	1.434690	1.685994	1.951762	2.230349	2.520066	2.819204	3.126052	3.438923	3.756172	4.076222
0.95	1.534117	1.802094	2.085186	2.381563	2.689343	3.006623	3.331498	3.662095	3.996595	4.333258
1.00	1.631464	1.915758	2.215793	2.529547	2.854943	3.189875	3.532242	3.879978	4.231084	4.583657