A NEW METHOD EVALUATION OF THE EFIE INTEGRAL FOR THE TWO-DIMENSIONAL DIRICHLET PROBLEM

Alexander G. Tyzhnenko*

*Department of Mathematics
Kharkov State Economic University
9-A Lenin Ave.Kharkov, Ukraine
E-mail: tyzhnenko@ksue.edu.ua

Abstract

A new method of direct integration (DIM) is proposed for evaluating integrals with integrable singularity. This method is applied for solving for 2-D E-scattering from metallic cylinders both for resonant and non-resonant cases. The Galerkin MoM with DIM is considered and compared to that of with singularity extraction approach. Applying DIM, a stable and effective solution to electric field integral equation (EFIE) is obtained for any scatterer configurations including thin elongated structures. A combined field integral equation (CFIE) is also considered with DIM. Making use of DIM and vectorized filling of MoM matrix in the MATLAB programming area, a high effective MoM solution to CFIE is obtained.

1. INTRODUCTION

The wide known work of Harrington [1] has made the MoM one of the most popular methods to numerical solution of integral equations in electromagnetics. However, the additional integration, in contrast to the collocation method, increases substantially the computational expenses of traditional MoM. When solving an EFIE, the cost of computations is increased additionally due to measures for suppressing the excessive sensibility of solutions to the meshing. The unstable behavior of an EFIE solution is caused by incorrect evaluation of logarithmically singular integral in this equation upon using quadrature formulas. This is connected with dependence of residual error on meshing length. Such an error to reduce, one has to increase the meshing density that results in additional computational cost. This problem to solve one may use the traditional method of singularity extraction [2], which is also known as the method of analytical regularization (AR). However, a sufficiently accurate evaluation of remained regular integral in AR takes much computational labor. In [3], this problem was considered in tree dimensions. In this paper, a new version of solving this problem in two dimensions is presented. For this purpose, a new method (DIM) is used, which permits to obtain a stable and effective solution to the Dirichlet problem. We consider here both the EFIE, which may be contaminated with interior-region resonances, and the CFIE, which have a unique solution. The latter equation results from the combined-field method [4,5]. Besides that, we consider the optimization problem to the Galerkin method of moments (GMoM) for solving both EFIE and CFIE in the MATLAB programming area. The vectorized filling of GMoM matrix coupled with high efficiency of DIM permits to obtain a numerical algorithm for CFIE

solver, which is more effective than that of point-matching (PM) solution to the magnetic field integral equation (MFIE).

The organization of the paper is as follows. In section II, the main equations are considered, which are solved in this paper with DIM, and the main approximate quantities are derived. In section III, the computational peculiarities are given. In section IV, the CFIE parameter is investigated with respect to accuracy and efficiency of GMoM solutions. In section V, the scattering from thick, thin, and very thin structures are investigated using elliptical cylinder with different axial ratios as a model. The solutions obtained with different methods are compared to investigate their accuracy and efficiency.

2. TWO-DIMENSIONAL DIRICHLET PROBLEM

Consider 2-D scattering of an E-polarized plane wave having only the z-component

$$E_0(x, y) = e^{-jk_1 x - jk_2 y - j\omega t} \tag{1}$$

where $k_1 = k \sin i_0$, $k_2 = k \cos i_0$, and i_0 is the incident angle counted clockwise from the negative y-axis. The time dependence is henceforth omitted. The wave (1) falls on a metallic cylinder of arbitrary cross section D having a smooth boundary ∂D. The total electric field is decomposed further into the incident and scattered fields as follows:

$$E = E_0 + E_{sc}$$

The scattered field E_{sc} must obey the scalar Helmholtz equation, radiation condition, and boundary condition

$$E_{sc} = -E_{0|\partial D}$$

The scattered field is sought as a linear combination of the single- and double-layer potential with some parameter η as in [4,5]

$$E_{sc} = \int_{\partial D} \frac{\partial G}{\partial n'} \varphi \, ds' - j\eta \int_{\partial D} G \varphi \, ds' \tag{2}$$

to prevent the resonant contamination of solutions of the CFIE

$$\frac{1}{2}\varphi + \int_{\partial D} \frac{\partial G}{\partial n'} \varphi \, ds' - j\eta \int_{\partial D} G \varphi \, ds' = -E_0 \tag{3}$$

Here G stands for the Green's function of the infinite space of wavenumber k. Because of the integral operators in (3) are compact on the space of continuous functions, the CFIE is a Fredholm second kind equation. The unicity of the solution of (3) was proved in [4] for $\eta > 0$ in tree dimensions but this proof is valid in two dimensions as well. We consider in this paper the numerical solution to (3) with the aid of GMoM on the set of $\{e^{jnt}\}_{n=-\infty}^{\infty}$ for the basis and test functions. We use further the natural parameterization of the boundary line

$$x = x^{(n)}(s), \quad y = y^{(n)}(s), \quad s \in [0, L] \tag{4}$$

where L is the length of the boundary curve. Then we pass to the unitless variable $t = 2\pi s / L$, $t \in [0, 2\pi]$. The unknown function φ is sought then as the Fourier expansion

$$\varphi(t) = \sum_{n=-\infty}^{\infty} a_n e^{jnt} \tag{5}$$

Standard treating yields the GMoM matrix equation

$$a_m + \sum_{n=-\infty}^{\infty} K_{mn} a_n = F_m, \quad m = 0, \pm 1, \ldots, \tag{6}$$

where

$$K_{mn} = \frac{L}{2\pi^2} \int_0^{2\pi} e^{-jmt} dt \int_0^{2\pi} P(t, t') e^{jnt'} dt' \tag{7}$$

$$P(t, t') = \frac{\partial G(kR)}{\partial n'} - j\eta G(kR) \tag{8}$$

$$R(t, t') = \sqrt{(x(t) - x(t'))^2 + (y(t) - y(t'))^2}$$

and

$$x = x(t), \quad y = y(t), \quad t \in [0, 2\pi]$$

is the parametric equation of the boundary ∂D. We also denote here the right-hand side as

$$F_m = -\frac{1}{\pi} \int_0^{2\pi} e^{-jmt - jk_1 x(t) - jk_2 y(t)} dt \tag{9}$$

and the Green's function as

$$G(kR) = \frac{1}{4j} H_0^{(2)}(kR) \tag{10}$$

Due to the compactness of integral operators in (3), both the matrix (7) and the right-hand side (9) are square summable in l_2 that ensures the existence and uniqueness of the solution to equation (6) and convergence of the approximate solution

$$\varphi_N(t) = \sum_{n=-N}^{N} a_n e^{jnt} \tag{11}$$

of its truncated counterpart

$$a_m + \sum_{n=-N}^{N} K_{mn} a_n = F_m, \quad m = 0, \pm 1, \ldots, \pm N \tag{12}$$

to the exact solution of integral equation (3) in Hilbert space of continuous functions. Let decompose the GMoM matrix (7) into two parts related to the MFIE and EFIE, respectively,

$$K_{mn} = K_{mn}^{(M)} - j\eta K_{mn}^{(E)} \tag{13}$$

where

$$K_{mn}^{(M)} = \frac{L}{2\pi^2} \int_0^{2\pi} e^{-jmt} dt \int_0^{2\pi} \frac{\partial G}{\partial n'} e^{jnt'} dt' \tag{14}$$

and

$$K_{mn}^{(E)} = \frac{L}{2\pi^2} \int_0^{2\pi} e^{-jmt} dt \int_0^{2\pi} G e^{jnt'} dt' \tag{15}$$

Then, one can write down the discretized counterpart of EFIE using the same notation

$$\sum_{n=-\infty}^{\infty} K_{mn}^{(E)} a_n = F_m, \quad m = 0, \pm 1, \ldots \tag{16}$$

At the same time, the analogous equation for the MFIE can be written as

$$a_m + \sum_{n=-\infty}^{\infty} K_{mn}^{(M)} a_n = F_m, \quad m = 0, \pm 1, \ldots \tag{17}$$

The equations (6), (16), and (17) are used for solving the same scattering problem but have different application ranges and numerical peculiarities. Optimization of their numerical algorithms and comparison of their solutions obtained with different techniques is the main goal of this work. An adequate and accurate solution to such an equation to obtain, one has to evaluate the matrix elements with high precision. Computation of $K_{mn}^{(M)}$ with the aid of simplest quadrature formula ensures a sufficient accuracy. As for $K_{mn}^{(E)}$, the magnitudes of these elements

are highly dependent on the method of computing the singular integral (15). Computational errors of these elements are crucial for the scattering problem solved. The diminishing of these errors with minimal growth of the total computational labor (TCL) is the main goal of our findings. Matrix elements $K_{mn}^{(E)}$ to evaluate, we use the Bogolyubov-Krylov method [6] of integrals discretization. Let represent (15) as

$$K_{mn}^{(E)} = \frac{L}{8\pi^2 j}\sum_{s=1}^{M}\int_{t_s-d}^{t_s+d} e^{-jmt}dt\sum_{p=1}^{M}\int_{t_p-d}^{t_p+d} H_0^{(2)}(kR(t,t'))e^{jnt'}dt' \tag{18}$$

and apply the abovementioned method. Then,

$$K_{mn}^{(E)} \approx \frac{Lh}{8\pi^2 j}\sum_{s=1}^{M} e^{-jmt_s}\sum_{p=1}^{M}\int_{t_p-d}^{t_p+d} H_0^{(2)}(kR(t_s,t'))e^{jnt'}dt' \tag{19}$$

We use here the uniform subdividing of $[0,2\pi]$ by M parts of the length $h = 2d = 2\pi / M$. The remained integral is evaluated in the simplest manner if $p \neq s$, and then we can write

$$\int_{t_p-d}^{t_p+d} H_0^{(2)}(kR(t,t'))e^{jnt'}dt' = \begin{cases} \approx hH_0^{(2)}(kR(t_s,t_p))e^{jnt_p}, & p \neq s \\ \kappa_{pn}, & p = s \end{cases} \tag{20}$$

where

$$\kappa_{pn} = \int_{t_p-d}^{t_p+d} H_0^{(2)}(kR(t_p,t'))e^{jnt'}dt' \tag{21}$$

Evaluation of this integral is the main point of our investigation. Along with this integral, we consider also its approximation

$$\tilde{\kappa}_{pn} = e^{jnt_p}\int_{t_p-d}^{t_p+d} H_0^{(2)}(kR(t_p,t'))dt' \tag{22}$$

to which one can apply the AR. The auxiliary function, which we use, is $\ln(ka|t_p - t'|)$. Adding and subtracting this function from integrand in (22), we obtain

$$\tilde{\kappa}_{pn} = e^{jnt_p}\left\{\{-4j(d\ln(kad)-d)/\pi\} + \int_{t_p-d}^{t_p+d}\{H_0^{(2)}(kR(t_p,t')) + 2j\ln(ka|t_p - t'|)/\pi\}dt'\right\} \tag{23}$$

The simplest approximation to this integral, which provides a minimal TCL, is

$$\tilde{\kappa}_{pn} \approx e^{jnt_p}\left\{-4j(d\ln(kad)-d)/\pi + 2d(1 - 2jC/\pi - 2j\ln(k\Lambda_p/2ka)/\pi)\right\} \tag{24}$$

where $\Lambda_p = (\dot{x}_p^2 - \dot{y}_p^2)^{1/2}$, and C is the Euler's constant. This approach yields in some cases an incorrect solution because of lack of accuracy, as will be demonstrated further. However, a more accurate analytical evaluation of the regular integral in (23) increases significantly the TCL. To overcome this problem, a new method of evaluating the integrals with integrable singularities was proposed in [3] for tree dimensional problems. In two dimensions, we propose the DIM for this purpose. In this method, an original singular integral is substituted by its nonsingular counterpart, which permits to apply a numerical (direct) integration. Let consider the integral (22). Its singularity to elimination, we use the analogous integrand for a circular cylinder. Then,

$$\tilde{\kappa}_{pn} = e^{jnt_p}\left\{2\int_0^d H_0^{(2)}(2ka\,|\sin\frac{\varsigma}{2}|)d\varsigma + \int_{t_p-d}^{t_p+d}\left\{H_0^{(2)}(kR(t_p,t')) - H_0^{(2)}(2ka\,|\sin\frac{t-t'}{2}|)\right\}dt'\right\} \quad (25)$$

The integrand in second integral is smoothly varying function and then it can be evaluated using the simplest quadrature formula. Then,

$$\tilde{\kappa}_{pn} \approx e^{jnt_p}\left\{2\int_0^d H_0^{(2)}(2ka\,|\sin\frac{\varsigma}{2}|)d\varsigma - \frac{2j}{\pi}\Lambda_p\right\} \quad (26)$$

Numerical investigations show a high accuracy of such an approach in contrast to the traditional AR (23, 24). Then, the main problem is in accurate evaluation of the remained integral in (26). Because this magnitude does not depend on index *p*, one needs to evaluate this integral only once. This may be done with the aid of different methods. In this work, we use DIM in which a singular integrand is substituted by its nonsingular counterpart, viz.,

$$\int_0^d H_0^{(2)}(2ka\,|\sin\frac{\varsigma}{2}|)d\varsigma \approx \int_0^d H_0^{(2)}(2ka\,|\sin\frac{\varsigma}{2}|+\varepsilon)d\varsigma \quad (27)$$

with very small ε. One can rigorously prove, using the finite part (FP) of a hyper-singular integral that absolute error of such a substitution may be estimated as $\varepsilon h^{-1/2}$. This error is negligible for used in this work magnitude of ε ($\varepsilon = 10^{-12}$) supposing any reasonable meshing length *h*. The resulted integrand is quite appropriate for numerical evaluation. The computing rate of integration code, which evaluates (27) in the GMoM, has strongly effected on the efficiency of the whole algorithm. Then, a special high-speed iterative integration routine has been done using the 14-point Gauss quadrature formula and vectorized operations in the MATLAB programming area. This code supports automatically a given tolerance level independently of the range of integration that eliminates completely the excessive sensibility of a solution to the meshing, and permits to obtain a stable and adequate solution to scattering

problem without enlarging the TCL when compared to the traditional AR with (24). The last point is the main advantage of the proposed method (DIM) and permits to increase drastically the efficiency of GMoM algorithm.

Worth mentioning is that the DIM may be implemented for solving not only the EFIE but also any other integral equations with weak singularity. In principle, there were no visible shortcomings for applying of this method in tree dimensions as well.

The proposed method (DIM) is used in this paper for solving both the EFIE and CFIE (3). In the first case, we investigate several solvers to the EFIE with DIM, and compare their accuracy and efficiency. In the second case, the main goal of our findings is the optimization of GMoM algorithm with DIM, and investigation of the applicability of CFIE for scattering from structures with thin cross section.

3. COMPUTATIONAL FEATURES

All computations have been done in the MATLAB programming area. The uniform meshing has been used in all discretization schemes according to the Bogolyubov-Krylov method [6]. The number of subintervals (n) was derived on the basis of m_f subintervals (points) per wavelength. The quantity m_f we call as 'meshing factor' and use throughout all computations while estimating the accuracy of numerical solutions comparing with the wide used rule of ten points (subintervals) per wavelength (m_f =10).

As the main feature of a numerical solution to the boundary value problem, we use a relative error of boundary conditions. Because the CFIE is a Fredholm second kind equation, the diminishing of discrepancy of an approximate solution indicates the approaching of a numerical solution to the exact one and may be used as a boundary condition error. Hence, we compute this error as

$$e_N = \left\| \frac{1}{2}\varphi_N + \int_{\partial D} \frac{\partial G}{\partial n'}\varphi_N ds' - j\eta \int_{\partial D} G\varphi_N ds' + E_0 \right\|_2 \backslash \| E_0 \|_2 \tag{28}$$

where φ_N is the (2N+1)-term approximation to an exact solution, and $\| \circ \|_2$ denotes the Euclidian norm. Such an error was computed on doubled meshing the maximal errors in midpoints to account. For the EFIE and MFIE the boundary condition errors were computed in analogous manner. The scattering cross section was used in unitless form as follows:

$$\sigma(\phi) = \lim_{\rho\to\infty} \frac{2\pi k\rho\,|E_{sc}|^2}{|E_0|^2} = 4\,|F(\phi)|^2 \tag{29}$$

where $F(\phi)$ is the scattering amplitude function. The monostatic scattering cross section (MRCS) was computed as usually, viz., $\sigma_b = \sigma(-i_0 - \pi/2)$. Because of comparing the approximate solutions for scattering from circular cylinders with the exact Rayleigh solutions in the far zone, we compute the corresponding matching error as

$$e_R = \| F_R(\phi) - F(\phi) \|_2 \,/\, \| F_R(\phi) \|_2 \tag{30}$$

where $F_R(\phi)$ is the scattering amplitude function of a Rayleigh solution. This error has been computed on uniform subdividing of interval $[0,2\pi]$ by 2000 parts. In all computations, we put the number of accounted Fourier harmonics equals to $N = [n/4]$, where brackets denote an integer part of a number, and n is the number of subintervals according to m_f points per wavelength. Such value of N is optimal, as to our investigation, to meeting the combined challenge of the speed and accuracy of computations. The most important feature of computations provided is the using of vectorized operations while filling GMoM matrices. Very high speed of vector procedures in the MATLAB programming area permits to diminish the TCL of GMoM algorithm so much that it approaches the one of collocation method. It is worthwhile to point out that using of natural parameterization while solving the MFIE both with MoM and collocation method reduces the computation cost by roughly a factor of 2.

We compare in this work several numerical methods solution of the Dirichlet problem in terms of efficiency and TCL. To more or less adequately comparison of different numerical algorithms, we use some particular definitions to such notions as the TCL and efficiency. To estimate the TCL independently from the computation power of a computer, we use a relative CPU time of computations

$$tcl = t_{CPU} \,/\, t_R(ka, N) \tag{31}$$

wherein a computation period is expressed in terms of the CPU time of a standard numeric algorithm. As such an algorithm, we use a well known procedure of the Rayleigh solution for the scattering of an E-polarized plane wave by a metallic circular cylinder with the same electric size *ka* that is used in considered problem, *a* being the radius of the minimum circumference that contains the scatterer, and with some definite number *N* of harmonics used in exact solution. The last quantity (*N*) is chosen so that the corresponding Rayleigh solution yields a boundary condition relative error of the order of 10^{-13}. In our investigations, we take *N* = 40 for *ka* = 10,

N = 100 for ka = 50, N = 200 for ka = 100, and N = 250 for ka = 150. The corresponding values of the MRCS are the following: 14.9957, 21.9622, 24.9718, and 26.7325.

To estimate the efficiency of a numerical algorithm, we use the following quantity:

$$eff = (e_N \cdot tcl)^{-1} \tag{32}$$

because it is clear from general considerations that the efficiency grows with the accuracy increasing and falls with the total computational labor increasing.

4. OPTIMIZATION OF THE CFIE

Utilization of both single- and double-layer potentials to prevent the irregular behavior of boundary value problem solutions due to spurious resonances of interior region was proposed by several authors at the same time (see [4]). In [4], the unicity of CFIE solutions was proved for $\eta > 0$. At the same time, in [7] such a proof have been done for any complex η. To solve the CFIE numerically, one must give some definite value to this parameter. It is clear a priori that η must not be chosen too small to prevent the contaminations from the MFIE eigen solutions. From the other side, η must not be chosen too large to prevent the contaminations from the EFIE eigen solutions. Besides that, the accuracy of MFIE solvers is substantially higher than that of EFIE with DIM for the same m_f regarding nonresonant cases. Because of that, high accuracy of CFIE solver to achieve η must be taken as small as possible. It is convenient to perform this parameter as $\eta = p/a$, where a is a half of maximal size of a scatterer. Such representation for η permits to find some optimal value of p independently of the scatterer size. The existence of optimal value of the CFIE parameter p is demonstrated in Table 1, where the boundary condition errors (e_N) of approximate solutions to the CFIE (3) for the circular cylinder of resonant electrical size, ka = 3.8317151, are tabulated for different values of p together with the errors in the far zone (e_R), MRCSs (σ_b), and GMoM matrix condition numbers (*cond*) for moderate value of meshing factor, m_f =15. The correspondent patterns in the far zone are shown in Figure 1, where we can see that, in general, we can diminish the parameter p to 0.004 without any response in the scattered field even in resonant points. To meeting the combined challenge of the accuracy in resonant and non-resonant regions, one can take p = 0.01 for scatterers with thick cross sections, and p = 0.4 for elongated structures.

Table 1 GMoM solutions with DIM to the CFIE for regressively smaller parameter p in the resonance case (ka = 3.8317151) for circular cylinder with m_f =15 (n = 57)

p	1	0.5	0.4	0.3	0.2	0.1	0.05	0.01
e_N	0.0063	0.0052	0.0049	0.0046	0.0043	0.0041	0.0041	0.0091
e_R	0.0066	0.0056	0.0053	0.0051	0.0049	0.0048	0.0051	0.0109
σ_b, dB	10.8810	10.9241	10.9359	10.4883	10.9607	10.9739	10.9851	11.0481
cond	3.9	7.8	9.7	13.0	19.5	39.1	78.3	398.0

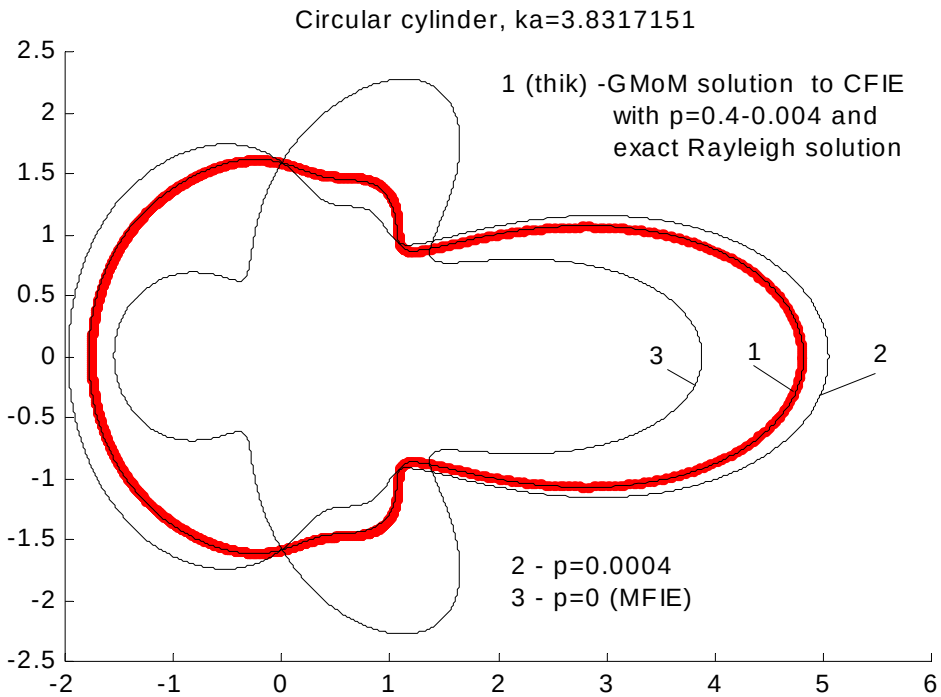

Figure 1. Far-field patterns for circular cylinder of resonant size resulted from solutions to the CFIE for various values of parameter p compared to the exact Rayleigh solution (thick curve).

At last, we briefly consider the influence of imaginary part of the CFIE parameter η upon accuracy of the GMoM solution to mentioned equation. Numerical investigations provided show the dependence of accuracy of a solution on the sign of imaginary part. Errors are greater for the negative one. In the contrary case, errors are the same as for real η if their absolute values are equal.

5. COMPARISON OF METHODS

This section has focused on GMoM solutions both to the EFIE itself with the traditional method of AR and with new one (DIM), and to the CFIE (3) with DIM. As was noted above, the utilization of AR with (24) in EFIE solvers can produce an unstable solution due to the lack of accuracy. Such case is considered in the first section (from above) of Table 2 for scattering from

circular cylinder of moderate electrical size (ka = 100) for progressively greater meshing factor. In the second section of Table 2, the solution of the same problem with DIM is given making use the approach (27). We see the solution become stable with respect to all main factors. Worth noting is that the implementation of direct integration (DIM) for improving the approach (24) does not increase practically the computational period when compared to AR with (24), which is the main advantage of a new method. Comparison with the exact Rayleigh solution in the far zone indicates the adequacy of GMoM solution to the EFIE with DIM.

Consider now the implementation of DIM in the CFIE solver with GMoM. We shall now show that using both DIM and vectorized filling of GMoM matrix in the MATLAB programming area permits to obtain a high effective GMoM solver for the CFIE, which is more effective than the point-matching (PM) algorithm for the MFIE. In Table 3 we compare this solution to the analogous solution of the MFIE with PM. Because of the scatterer has thick cross section, we use the CFIE with p = 0.01. It is clear that the TCL of CFIE with GMoM must be greater than that of MFIE with PM for the same m_f but the exceeding is not large as we can see from Table 3. At the same time, the accuracy of CFIE solver with GMoM is substantially higher than that of MFIE with PM, that provides a substantially higher efficiency of GMoM with DIM.

Table 2 GMoM solutions to the EFIE with AR and DIM for circular cylinder of ka = 100

$m_f\,(n)$		4 (400)	5 (500)	6 (600)	7 (700)	8 (800)	9 (900)	10 (1000)
GMoM with AR	e_N	0.0232	0.0298	0.0741	0.0322	0.0286	0.0740	0.0198
	e_R	0.0266	0.0307	0.0667	0.0307	0.0331	0.0800	0.0216
	σ_b, dB	24.7125	24.5189	24.0653	24.4378	24.4324	25.5477	25.0456
	cond	$1.0\cdot10^3$	$2.9\cdot10^3$	$1.0\cdot10^4$	$3.5\cdot10^3$	$3.0\cdot10^3$	$1.1\cdot10^3$	$2.4\cdot10^3$
	tcl	3.3	5.3	8.6	12.6	18.1	23.9	31.1
	eff	13.1	6.3	1.6	2.5	1.9	0.56	1.6
GMoM with DIM	e_N	0.0486	0.0372	0.0314	0.0260	0.0235	0.0248	0.0256
	e_R	0.0553	0.0417	0.0347	0.0286	0.0253	0.0257	0.0274
	σ_b, dB	25.2103	25.0693	24.8578	24.8522	24.8680	24.7335	24.6864
	cond	168	352	441	602	1101	1725	2600
	tcl	3.6	5.8	8.7	13.4	16.7	23.8	33.6
	eff	5.7	4.6	3.7	2.9	2.6	1.7	1.2

Table 3 GMoM solutions with DIM to the CFIE (p = 0.01) and PM solutions to the MFIE via meshing factor for circular cylinder of ka = 100

$m_f\,(n)$		4 (400)	5 (500)	6 (600)	7 (700)	8 (800)	9 (900)	10 (1000)
CFIE, GMoM with DIM	$e_N \cdot 10^4$	11.00	1.98	0.188	0.172	0.157	0.144	0.132
	$e_R \cdot 10^4$	12.02	5.67	3.15	1.94	1.29	0.90	0.657
	σ_b, dB	24.9785	24.9753	24.9738	24.9730	24.9726	24.9723	24.9722
	cond	41.3175	41.3186	41.3186	41.3184	41.3182	41.3180	41.3178
	tcl	6.6	10.2	16.1	23.0	33.3	45.4	57.5
	eff	137.7	495.2	3303.8	2527.8	1912.7	1529.6	1317.5
MFIE with PM	$e_N \cdot 10^4$	12.03	5.51	3.05	1.86	1.23	0.851	0.615
	$e_R \cdot 10^4$	12.31	5.72	3.15	1.93	1.28	0.887	0.642
	σ_b, dB	24.9786	24.9754	24.9739	24.9731	24.9727	24.9724	24.9722
	cond	41.3136	41.3150	41.3155	41.3157	41.3158	41.3159	41.3159
	tcl	3.6	6.6	10.6	16.3	24.3	34.0	42.3
	eff	231.5	275.5	304.3	329.8	334.6	346.0	384.3

Let consider now scattering from elongated structures. For example, consider an elliptical cylinder of moderate electrical size, ka = 100, with axial ratio b/a = 0.01. In this case, the MFIE itself gives no correct solutions but the CFIE with p = 0.4 gives still quite correct ones. GMoM solutions with DIM to the CFIE for mentioned scatterer are given in the first section (from above) of Table 4 for oblique incidence ($i_0 = 45^\circ$). We can see here that the efficiency of GMoM solver for the CFIE is substantially lower with p = 0.4 then that of with p = 0,01.In the second sections of the same table, we give GMoM solutions to the EFIE with DIM. The comparison of solutions indicates the adequacy of EFIE solvers because of solutions to the CFIE have a guaranteed convergence. In spite of the less accuracy of GMoM solution to the EFIE when compared to that of the CFIE, the corresponding scattering amplitude functions are not distinguishable for used in Table 4 magnitudes of m_f if using p = 0.4.

Some discrepancy between solutions to demonstrate, we show in Figure 2 solutions of the CFIE intentionally with not optimal value of p (p = 0.1) compared to analogous solutions to the EFIE, all with DIM, for the same scattering problem and small meshing factor (m_f = 6). Diminution of parameter p in the CFIE causes the instability of solutions for thin scatterers. Such incipient instability we can see in Figure 1 (a, d) near observation direction of $\phi = \pi$. However, in general, the overlapping of solutions is rather good even for this bad case.

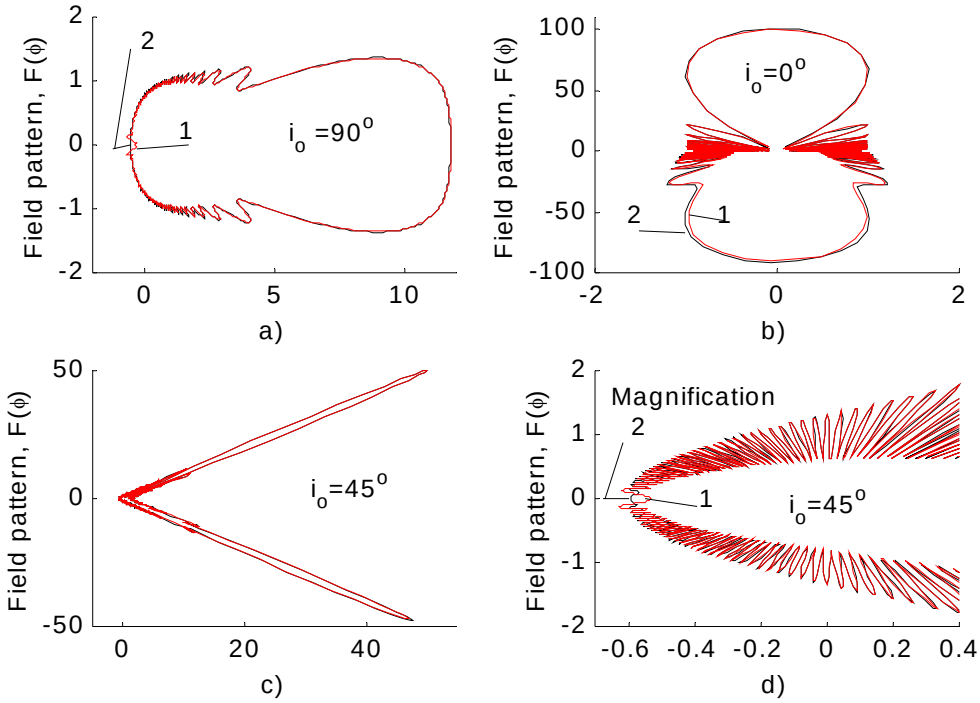

Figure 2. Far-field patterns for scattering from the elliptical cylinder of *ka* = 100 with axial ratio *b/a* = 0.01 obtained by solving the CFIE with intentionally small *p* = 0.1 that is not optimal in this case and the EFIE, both with DIM. We can see the incipient instability of CFIE solver (curves 1) for this *p* near observation direction of $\phi = \pi$. For *p* = 0.4, these two solutions give identical patterns.

Table 4. GMoM solutions with DIM to the CFIE and EFIE for elongated elliptical cylinder with axial ratio *b/a* = 0.01 (oblique incidence, $i_0=45^o$)

$m_f\,(n)$		7 (444)	8 (508)	9 (572)	10 (636)	11 (700)	12 (764)
CFIE, GMoM with DIM	$e_N \cdot 10^4$	4.17	3.85	3.54	3.26	3.01	2.79
	σ_b, dB	1.6003	1.5843	1.5793	1.5774	1.5766	1.5760
	cond	19.6955	19.6712	19.6625	19.6585	19.6560	19.6542
	tcl	9.6	13.5	17.6	24.4	31.9	41.9
	eff	249.8	192.4	160.5	125.7	104.2	85.5
EFIE, GMoM with DIM	e_N	0.0250	0.0219	0.0194	0.0173	0.0157	0.0143
	σ_b, dB	1.6179	1.6119	1.6072	1.6035	1.6004	1.5979
	cond	50	57	64	71	78	85
	tcl	3.8	5.2	7.0	10.1	12.4	16.3
	eff	10.5	8.8	7.4	5.7	5.2	4.3

Table 5. GMoM solutions with DIM to the EFIE for elliptical cylinder of *ka* = 100 and axial ratio *b/a* = 0.001 (oblique incidence, i_0=45º)

$m_f(n)$		10 (636)	11 (700)	12 (762)	13 (826)	14 (890)	15 (954)
EFIE, GMoM with DIM	e_N	0.0601	0.0332	0.0242	0.0194	0.0164	0.0144
	σ_b, dB	0.2682	0.2239	0.2105	0.2049	0.2024	0.2011
	cond	2000	761	515	411	358	326
	tcl	9.8	13.4	17.2	19.9	26.0	32.4
	eff	1.7	2.3	2.4	2.6	2.4	2.2

Table 6. GMoM solutions with DIM to the EFIE for elliptical cylinder of *ka* = 50 with axial ratio *b/a* = 0.001 (oblique incidence, i_0=45º)

$m_f(n)$	20 (636)	21 (668)	22 (700)	23 (732)	24 (762)	25 (794)
e_N	0.0256	0.0189	0.0152	0.0129	0.0114	0.0102
σ_b, dB	3.0315	3.0312	3.0314	3.0317	3.0321	3.0325
cond	1621	1049	805	671	591	532
tcl	9.3	10.6	12.0	12.8	14.6	16.3
eff	4.2	4.5	5.5	6.1	6.0	6.0

Table 7. GMoM solutions with DIM to the EFIE for elliptical cylinder of *ka* = 150 with axial ratio *b/a* = 0.001 (oblique incidence, i_0=45º)

$m_f(n)$	6 (572)	7 (668)	8 (762)	9 (858)	10 (954)	11 (1050)
e_N	$9.5\cdot10^3$	0.0701	0.0382	0.0276	0.0223	0.0189
σ_b, dB	83.6507	3.0362	3.0355	3.0365	3.0374	3.0381
cond	$2.3\cdot10^8$	635	353	271	236	217
tcl	7.3	10.2	14.3	19.8	27.5	34.8
eff	-	1.4	1.8	1.8	1.6	1.5

At last, we consider a scattering from very thin structures using an elliptical scatterer with axial ratio equal to 0.001 as a model. Such problems arise, for example, in investigations connected with radar returns from rough surfaces. The main scattering features for this case resulted from solutions to the EFIE with DIM are listed in Table 5 for *ka* = 100 and $i_0 = 45^\circ$. With the meshing factor progressively increasing, the condition number increases to sufficiently high value, as it is usual for the EFIE solutions, but then it exhibits a monotonously decrease. We show such decreasing in Table 5 for the GMoM solution to the EFIE. The same decreasing was also revealed in analogous solution with the Galerkin collocation, which is not given here for brevity. Because of high interest of this effect, as to our mind, we give here another two GMoM solutions with DIM via m_f for elliptical scatterers with the same *b/a* = 0.001 but with others electrical sizes: for *ka* = 50, in Table 6 and for *ka* = 150, in Table 7. The analogous behavior of the condition number of GMoM matrix while solving the EFIE with DIM was revealed for *b/a* = 0.0001 as well. This effect permits to consider the obtained solutions as good

candidates to exact solutions. In any case, the diminishing of condition number ensures the approaching of approximate solution to an exact one if the corresponding discrepancy of the matrix equation diminishes. This point is of principle significance if we solve an integral equation of the first kind. The revealed behavior of GMoM solution to the EFIE for very thin structures permits to simulate large areas of rough surfaces and obtain stable solutions to corresponding Dirichlet problems.

6. CONCLUSIONS

A new method of direct integration (DIM) has been presented and applied to solve the EFIE and CFIE with the Galerkin method of moments (GMoM). When compared to the simplest version of analytical regularization (AR), which is also called as “singularity extraction”, DIM gives much more accurate solution with the same computational labor. GMoM solver with DIM permits to obtain a stable, adequate, and high effective solution to the EFIE for any scatterer configuration. Analyzing scattering from very elongated elliptical cylinders, which axial ratios are less or equal to 0.001, a new effect has been revealed: the condition number of GMoM matrix decreases with the meshing density increasing. Such falling of the condition number is unusual for the EFIE solvers. However, wide numerical investigations have been confirmed the existence of this effect in all investigated scattering problems connected with very thin structures. Such a behavior of the condition number while solving the EFIE ensures approaching of an approximate solution to the exact one if the matrix equation discrepancy recedes to zero.

Employing DIM and using vectorized operations in the Matlab programming area while filling GMoM matrices when solving the CFIE, a high effective numerical algorithm has been obtained, which is more effective than that of the MFIE with point matching. The obtained algorithm is valid for thin structures as well, if the minimal to maximal size ratio of a scatterer is more or equal to 0.01.

REFERENCES

[1] R. F. Harrington, *Field Computation by Moment Method*, NY, Macmillan, 1968.

[2] V. Sladek and J. Sladek, Eds., *Singular Integrals in Boundary Element Method*, Southampton, U.K.: Computational Mechanics Publications, 1998.

[3] D. J. Taylor, “Accurate and Efficient Numerical Integration of Weakly Singular Integrals in Galerkin EFIE Solutions”, *IEEE Trans. Antennas Propagat*., Vol. AP-51, No. 7, 2003, pp. 1630- 1637.

[4] D. Colton and R. Kress, *Integral Equations Methods in Scattering Theory*, NY, Wiley, 1983.

[5] N. Morita, N. Kumagi, and J. R. Mautz, *Integral Equations Methods for Electromagnetics*, Boston-London, Artech House, 1990.

[6] L. V. Kantorovich and V. I. Krylov, *Approximate Methods of Higher Analyses*, 4th edition, New York, Wiley, 1964.

[7] O. I. Panich, “*To the Solvability of Exterior Boundary Problems for the Wave Equation and Maxwell’s Equations*”, *Russian Mathematical Surveys (Engl.Transl.)*, Vol. 20, No. 1, 1965, pp. 221-226.