AN ACCURATE AND EFFICIENT METHOD OF SOLUTION OF THE ELECTRIC FIELD INTEGRAL EQUATION IN TWO DIMENSIONS

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Abstract

A new method evaluation the electric field integral equation integral is proposed that is based on analytic evaluation of this integral in the vicinity of singular point. This new method is compared to the traditional method of singularity extraction in terms of accuracy and efficiency for solving the Dirichlet problem in the integral equation formulation making use of the Galerkin method of moments. A high accuracy and efficiency of the new method is demonstrated both for the combined field integral equation and electric field integral equation solvers.

1. INTRODUCTION

In this work, a new method evaluation of singular electric field integral equation (EFIE) integral is proposed substantially more accurate than that of the traditional singularity extraction (SE) method [1]. The latter is the most accurate compared to other existed methods. Its simplest version with remained integral evaluated in the Bogoluybov-Krylov [2] approach is the most economical but not sufficiently accurate. In principle, it is possible to improve this simplest version but it takes much computational labor. To solve this problem in three dimensions, a new approach was proposed in [3]. In presented work, this problem is considered in two dimensions. A method analytical evaluation (AE) of the EFIE integral in the vicinity of singular points is proposed in this work. The main merit of this method is the substantial increasing of accuracy without increasing of the total computational labor (TCL) compared to simplest version of the SE method.

2. EVALUATION OF THE EFIE INTEGRAL

For closed boundary, the EFIE operator can be represent as

$$\int_0^{2\pi} H_0^{(2)}(kR(t_m,t'))\psi(t')dt' = \sum_i \int_{t_i-d}^{t_i+d} H_0^{(2)}(kR(t_m,t'))\psi(t')dt' \qquad (1)$$

where $t_m \in [0,2\pi]$, $R(t_m,t') = \sqrt{(x(t_m)-x(t'))^2 + (y(t_m)-y(t'))^2}$ and $x = x(t), y = y(t)$ is a parametric equation of the boundary. Here, $H_0^{(2)}(\cdot)$ is the zero order Hankel function of the second kind, k is a free space wave number, $2d = h$ is the grid spacing ($h = 2\pi / n$), and $\psi(t)$ is an unknown function proportional to a surface current. For a smooth boundary, this function is continuous and can be moved outside the integral in the midpoint $t' = t_i$. The remained integral is nonsingular for $t_i \neq t_m$ and can be computed with guaranteed accuracy. The main error produces evaluation of the singular EFIE integral for $t_i = t_m$:

$$J_i = \int_{t_i-d}^{t_i+d} H_0^{(2)}(kR(t_i,t'))dt' \tag{2}$$

The known SE method evaluation of this integral makes use of logarithmic function

$$H(t_i - t') = -\frac{2j}{\pi}\ln(ka\,|t_i - t'|) \tag{3}$$

as a regularizer. After singularity extraction, we obtain

$$J_i \approx -\frac{2j}{\pi}(\ln(kad) - 1 + \ln(L_i / 2a)) + (1 - 2jC/\pi)h \tag{4}$$

in the simplest version. Here, $L_i = \sqrt{\dot{x}^2(t_i) + \dot{y}^2(t_i)}$, C is the Euler's constant (C=0.577…), and a is a half of maximal scatterer size. In contrast to the SE method, we use in this work the following function

$$G(t_i - t') = H_0^{(2)}(2ka\,|\sin(t_i - t')/2|) \tag{5}$$

as a regularizer. Then,

$$\begin{aligned} J_i &= 2\int_0^d H_0^{(2)}(2ka\sin(\varsigma/2))d\varsigma + \int_{t_i-d}^{t_i+d}\{H_0^{(2)}(kR(t_i,t')) - G(t_i - t')\}dt' \\ &\approx 2\int_0^d H_0^{(2)}(2ka\sin(\varsigma/2))d\varsigma - \frac{2j}{\pi}\ln(kL_i) \end{aligned} \tag{6}$$

The remained singular integral can be evaluated using the known identity

$$\int_0^{2\pi} H_0^{(2)}(2ka\,|\sin(t-t')/2|)e^{jnt'}dt' = 2\pi J_n(ka)H_0^{(2)}(ka)e^{jnt}$$

as follows:

$$\int_0^d H_0^{(2)}(2ka\sin(\varsigma/2))d\varsigma = 2\pi J_0(ka)H_0^{(2)}(ka) - \int_d^{2\pi} H_0^{(2)}(2ka\sin(\varsigma/2))d\varsigma \tag{7}$$

Now, the remained integral in (7) is nonsingular and can be evaluated with guaranteed accuracy.

In order to compare the accuracy and efficiency of the new method proposed with those of the traditional SE method, we consider the scattering from a metallic cylinder with the aid of the Galerkin MoM (GMoM) on the set of $\{e^{jnt}\}_{n=-\infty}^{\infty}$. To this end, we consider the combined potential method [5] of electric field representation

$$E = E_0 + \int_{\partial D} \frac{\partial G}{\partial n'} \psi \, ds' - j\eta \int_{\partial D} G\psi \, ds' \tag{8}$$

that leads to the combined field integral equation (CFIE)

$$\frac{1}{2}\psi + \int_{\partial D} \frac{\partial G}{\partial n'} \psi \, ds' - j\eta \int_{\partial D} G\psi \, ds' = -E_0, \tag{9}$$

all in familiar notation. This equation contains the EFIE operator and we will derive the corresponding EFIE integral with the aid of two mentioned methods. When solving the equation (9), we put $\eta = p/a$. For not very elongated structures, the parameter p has been taken as 0.01 but for thin structures, it has been taken as 0.4 to meeting a combined challenge of accuracy and TCL.

3. COMPUTATIONAL FEATURES

The relative boundary error e_N was computed as the Euclidean norm of residual error for (2N+1)-terms approximation to an exact solution divided by $\| E_0 \|_2$ versus progressively greater value of the grid resolution $g_r = \lambda / h$ (a number of subintervals or points per wavelength) with h the grid spacing. The latter parameter, g_r, was proposed in [5] instead of the number of subintervals (n) and has, to our opinion, an insightful physical meaning. While using the GMoM, the truncation number N is related with n as $N = [n/4]$, to support the needed accuracy while evaluating the peripheral MoM matrix elements. The number (n) of subintervals, which are used for approximate evaluation of the MoM matrix elements, is

derived from the formula: $n = [|\partial D| kag_r / 2\pi]$, where $|\partial D|$ is the boundary length and $[\cdot]$ is the integer part of a number. To compare different numerical algorithms in terms of efficiency and TCL, we estimate these notions as numeric quantities. TCL can be estimated as the ratio of CPU time, t_{CPU}, of tested algorithm to that of any standard numerical one. As such a standard algorithm, we use the (2*N*+1)-terms approximation to well known Rayleigh solution for scattering of a E-polarized plane wave by a circular metallic cylinder of the same electric size (*ka*). The CPU time of this standard algorithm we denote as $t_R(ka, N)$. Then, TCL can be estimated numerically as

$$tcl = t_{CPU} / t_R(ka, N). \tag{10}$$

For *ka* = 100, we take *N* = 200, and for *ka* = 150, *N* = 250. To estimate efficiency numerically, we use a common economic definition, that of the accuracy per unit TCL. Then, the efficiency can be estimated as

$$eff = (e_N \cdot tcl)^{-1} \tag{11}$$

Such estimations of the main characteristics of an algorithm, permits one to compare different numerical methods independently from theirs machinery realization. Besides these characteristics, we also use a monostatic RCS (MRCS) in unitless form

$$\sigma_b = \lim_{\rho \to \infty} (2\pi k\rho |E_{sc}|^2 / |E_0|^2).$$

4. NUMERICAL RESULTS

Using all mentioned characteristics, we compare in Table 1 solutions to the CFIE with GMoM for circular cylinder making use of the SE and AE methods to each other, and both of these to the solution of the MFIE with point-matching (PM) method versus progressively greater g_r. Despite the lack of accuracy giving by SE method while solving the EFIE, its implementation in the CFIE solver gives sufficiently good solution as we can see from the first part (from top) of Table 1. In comparison, the exact value of MRCS derived from Rayleigh solution is $\sigma_b = 24.9718$. The analogous solution with the aid of new method is demonstrated in the second part of Table 1. Of the advantages of new method, the boundary error decreases more rapidly with g_r growth for the same TCL. This leads to efficiency increasing in contrast to the SE method. In the third part of Table 1, we collect the

characteristics of solutions to the MFIE with PM. Worthwhile to note only a little difference in the TCL between solutions with GMoM and PM (~ 35 %). This is due to using very fast vector operations of MATLAB programming area while filling the MoM matrix. Besides that, using a complete orthonormal set of basis and test functions increases significantly accuracy of GMoM solutions compared to PM ones. As a result, GMoM solutions to the CFIE are about 15 times more effective than those of the MFIE with PM.

Table 1 Scattering characteristics for circular cylinder. *ka* = 100. Exact monostatic RCS: σ_b =24.9718.

g_r		5	6	7	8	9	10
CFIE with SE	$e_N\ 10^4$	0.1977	0.1877	0.1727	0.1577	0.1441	0.1322
	σ_b	24.9755	24.9740	24.9732	24.9727	24.9724	24.9723
	tcl	8.9	14.1	21.0	29.6	41.9	55.4
	eff 10^{-3}	5.7	3.8	2.8	2.1	1.7	1.4
CFIE with AE	$e_N\ 10^4$	0.2213	0.1543	0.1089	0.0762	0.0516	0.0325
	σ_b	24.9754	24.9739	24.9731	24.9727	24.9724	24.9722
	tcl	9.0	14.3	21.6	29.8	41.7	56.3
	eff 10^{-3}	5.0	4.5	4.3	4.4	4.6	5.5
MFIE with PM	$e_N\ 10^4$	5.5107	3.0531	1.8617	1.2341	0.8511	0.6151
	σ_b	24.9754	24.9739	24.9731	24.9727	24.9724	24.9722
	tcl	6.6	10.6	16.3	24.3	34.1	42.3
	eff 10^{-3}	0.27	0.31	0.32	0.33	0.34	0.38

In spite of the CFIE yields a unique solution in contrast to the EFIE, and its solution is much more accurate, practitioners use, as a rule, the EFIE in design works due to a less TCL. However, one has to account a great difference between accuracies of theirs solutions, which leads to substantially larger efficiency of CFIE solutions compared to those of the EFIE. Along with this, the use of fast vector operations of the MATLAB programming area while filling MoM matrices yields practically the same TCL for both equations. To demonstrate full advantages of CFIE solutions, we collect in Table 2 the characteristics of the mentioned scattering problem solved with the aid of the EFIE. In the first part (from top) of this table, we give the characteristics derived with the aid of the new method (AE) evaluation of the EFIE integral. Comparing these ones with those of the CFIE solutions in Table 1, we can see dramatic difference in accuracy. When using the discretization of 10 points per wavelength ($g_r = 10$), the CFIE solution is about 10^3 times more accurate than the corresponding

solutions of the EFIE but with practically the same TCL. This leads to ~10^3 times larger efficiency of the CFIE solutions compared to the EFIE ones. In the second part of Table 2, we give the characteristics of analogous solution with the SE method with the goal to show that the traditional method (SE) evaluation of the EFIE integral sometimes gives unstable solutions. This occurs, as a rule, for $g_r < 10$. For larger g_r the solution becomes stable and the condition number (*cond*) decreases tending to its theoretical value. The same behavior of the *cond* observes also when using the new method (SE), as we can see from the first part of Table 2. However, in this case no instability occurs. In the common range of stability ($g_r > 10$), implementation of the new method (AE) gives substantially larger accuracy and efficiency than that of the SE. It is worthwhile to note that significant increasing of the *cond* for small g_r occurs due to large error evaluation of matrix elements. Then, the condition number of MoM matrix is the less the accurate is the method evaluation of the EFIE integral. The condition number diminishing with g_r growth when $g_r > 10$ does not mean its boundness. According to theoretical predictions, this diminution will vanish for some g_r and then the condition number will increase.

Despite the CFIE solutions are much more effective than the EFIE ones, in some cases one has to solve only the latter equation. Example is the scattering from very elongated structures, in which case the CFIE is not valid. In the connection with this issue, the principal importance has the method evaluation of the EFIE integral, which determines the accuracy and efficiency of BVP solutions. To demonstrate the advantages of proposed method (AE) in terms of the accuracy, TCL, and efficiency, we give in Table 3 the scattering characteristics for very elongated elliptical cylinder of $ka = 150$ and $b/a = 10^{-3}$, where b/a is the axes ratio (the incidence angle is $i_0 = 45^\circ$). Analogously to the circular cylinder, the condition number sharply increases with maximum at $g_r = 6$ and then decreases to corresponding theoretical value. For $g_r > 6$, both methods yield stable solution but the new method is more accurate and efficient. Analogously to the circular cylinder, the efficiency of BVP solution with the new method increases with g_r growth in contrast to that of with the SE.

Table 2. Scattering characteristics for circular cylinder of ka = 100 as the solutions of the EFIE.

g_r		5	6	7	8	9	10	11	12
EFIE, GMoM with AE	e_N	0.0245	0.0172	0.0120	0.0084	0.0059	0.0036	0.0019	0.0008
	σ_b	25.0342	24.9791	24.9968	24.9819	24.9947	25.0002	24.9900	24.9848
	cond	340	598	605	997	1980	1350	871	661
	tcl	4.5	6.8	9.9	14.9	20.8	28.1	37.8	50.1
	eff	9.1	8.5	8.4	8.0	8.2	9.9	13.9	23.8
EFIE, GMoM with SE	e_N	0.0298	0.0741	0.0322	0.0286	0.0740	0.0198	0.0149	0.0129
	σ_b	24.5189	24.0653	24.4378	24.4324	25.5477	25.0456	24.9746	24.9522
	cond	$2.9\ 10^3$	$1.0\ 10^4$	$3.5\ 10^3$	$3.0\ 10^3$	$1.1\ 10^4$	$2.4\ 10^3$	1374	1000
	tcl	5.3	8.6	12.6	18.1	23.9	31.1	39.7	47.5
	eff	6.3	1.6	2.5	1.9	0.6	1.6	1.7	1.6

Table 3. Scattering from elongated cylinder of ka =150, b / a = 10^{-3}, $i_0 = 45^\circ$.

g_r		5	6	7	8	9	10	11	12
EFIE, GMoM with AE	e_N	0.0568	0.1685	0.0333	0.0171	0.0105	0.0068	0.0043	0.0027
	σ_b	3.0684	3.2748	3.0745	3.0480	3.0416	3.0391	3.0379	3.0371
	cond	259	2108	491	296	231	202	186	177
	tcl	3.4	5.5	8.1	11.6	16.0	21.7	29.1	34.8
	eff	-	-	3.7	5.1	6.0	6.8	8.0	10.6
EFIE, GMoM with SE	e_N	0.0152	0.0702	0.0294	0.0221	0.0185	0.0163	0.0146	0.0134
	σ_b	3.0009	3.1076	3.0365	3.0356	3.0366	3.0375	3.0381	3.0386
	cond	381	$3.3\ 10^5$	576	339	267	233	216	206
	tcl	3.4	5.4	8.0	12.2	16.4	21.1	28.2	38.5
	eff	-	-	5.7	3.7	3.3	2.9	2.4	1.9

5. CONCLUSION

A substantial advantage of the new method (AE) evaluation the EFIE integral has been shown when solving the Dirichlet problem both in the CFIE and EFIE formulations compared to the traditional method (SE). To this end, numerical estimations of the total computational labor (TCL) and efficiency were done, which permit to compare different methods solution of the same boundary-value problem with the goal to reveal the most economical and accurate method. Using these estimations, it was shown the GMoM solution to the CFIE to be much more effective than that of the MFIE with PM method in the range of correctness of the latter equation. The EFIE solutions have been also estimated in terms of accuracy, TCL, and efficiency for very elongated structures, for which both the CFIE and MFIE are not apropos.

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