CONVERGERNT GALERKIN MOM SOLUTION TO THE HYPERSINGULAR 1ST KIND INTEGRAL EQUATION FOR 2-D NEUMANN PROBLEM

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Abstract

Two-dimensional solution to the Neumann problem is obtained in the 1st kind hypersingular integral equation (HSIE) formulation with the aid of Galerkin method of moments (GMoM) on the complete orthogonal basis. Convergence of the GMoM and uniqueness of the HSIE solution are proved in the Hilbert space of square integrable functions (L_2) for a smooth closed boundary of a scatterer in the nonresonance case. The exact relation between actual and residual errors is also obtained, which permits to calculate an actual error by integrating of residual error along the boundary of a scatterer.

1. INTRODUCTION

Two-dimensional problem of H-polarized plane wave scattering by an arbitrarily shaped metallic cylinder with closed smooth boundary is addressed. As is known, this problem can be considered with the aid of different methods. The most convenient, as to our opinion, is the integral equation formulation. The known method of the magnetic field integral equation (MFIE) solving gives excellent result for thick scatterers, but for thin ones and screens, it is not apropos. At the same time, scattering from very elongated structures and screens is of paramount importance when investigating radar returns from rough surfaces and wing-like scatterers, for example. Such problems can be considered in integral equation formulation with the aid of HSIE only. In principle, one can solve the HSIE analytically for thick scatterers with smoothed closed boundary [1]. However, in this case the MFIE is more apropos. There are known numerical methods, which used approximate evaluation of hypersingular integrals as the finite part in the sense of Hadamard [2]. These methods are applicable both for thin scatterers and screens but obtained solution depends on the closeness to the singular point and can be scarcely regarded as rigorous. Method solution of the HSIE developed in [3] with the aid of numerical evaluation of hypersingular integrals with the use of quadrature formulas can be regarded as approximate as well. Theoretically, the existence and uniqueness of the HSIE solution was considered in [4] for arbitrarily shaped

scatterers in full wave formulation making use of energy Sobolev space. However, the problem obtaining of mathematically correct solution convenient for practical using is not solved to now. This paper will attempt to solve this problem in two dimensions. We consider herein the scatterers with closed smooth boundary leaving screens for the next paper. The HSIE is solved in this work with the aid of the GMoM in the familiar Hilbert space L_2. To this end, the regularization of HSIE is fulfilled with the aid of special regularizer, which can be evaluated analytically for basis functions used as the finite part in the sense of Hadamard. This one permits us to transform the HSIE to a 2nd kind equation in the space of sequences. The convergence of this equation' solution is proved in the Hilbert space of sequences for nonresonant case, which proves the convergence of the GMoM in L_2. An important problem estimation of an actual error' norm in L_2 is also considered in this paper. The completeness of basis functions used permits us to obtain an analytic relation between actual and residual error norms in L_2 space. The obtained result permits one to avoid the using of the more complicated Sobolev space when computing actual errors as has been proposed in [4]. Besides that, it is worthwhile to point out the practical merits of proposed method solution to the HSIE. It has sufficiently high accuracy and the same total computational labor (TCL) as the MFIE with GMoM solver. However, in contrast to the last one the new method is valid both for thin scatterers and screens.

At last, the new method is compared to the known methods solution to the Neumann problem with the goal of estimating its accuracy and efficiency. Namely, it is compared to the Rayleigh solutions for circular cylinder and to the MFIE solutions for not very elongated cylinders. The scattering features of very elongated cylinder are also considered.

2. GALERKIN METHOD OF MOMENTS

Let decompose the total field as the sum of incident and scattered fields. The incident field

$$H_0(\vec{r}) = e^{-ikx\sin i_0 - iky\cos i_0}$$

impinges on the two-dimensional object with incidence angle i_0 counted clockwise from the mines Y direction. Here (x, y) is the observation point and k is the free space wavenumber. The scattered field is sought as a double-layer potential

$$H_s(\vec{r}) = \int_{\partial D} \frac{\partial}{\partial n'} G(\vec{r},\vec{r}\,')\varphi(\vec{r}\,')dl' \tag{1}$$

where

$$G(\vec{r},\vec{r}\,') = \frac{1}{4i} H_0^{(2)}(k\,|\,\vec{r}-\vec{r}\,'|)$$

stands for the two-dimensional Green's function, and $\varphi(\vec{r})$ is an unknown function, which is proportional to the surface current. The Neumann condition yields the integrodifferential equation

$$\frac{\partial}{\partial n} \int_{\partial D} \frac{\partial}{\partial n'} G(\vec{r},\vec{r}\,')\varphi(\vec{r}\,')dl' = F(\vec{r}) \tag{2}$$

(2)

with the right-hand side (RHS)

$$F(\vec{r}) = -\frac{\partial}{\partial n} H_0(\vec{r})\Big|_{\partial D} \quad . \tag{3}$$

Let us take the normal derivative in (2) inside the integral understanding the resulted integral as the finite part in the sense of Hadamard and representing the boundary ∂D in parametric form as $\vec{r} = \vec{r}(t), t \in \Omega, \Omega = [0;2\pi]$. Then the equation (2) becomes

$$\mathrm{FP} \int_0^{2\pi} \frac{\partial^2 G(\vec{r},\vec{r}\,')}{\partial n \partial n'} u(t')L(t')dt' = \widetilde{F}(t),\ t \in [0;2\pi], \tag{4}$$

where $u(t) = \varphi(\vec{r}(t)), \widetilde{F}(t) = F(\vec{r}(t)), L(t) = |\dot{\vec{r}}(t)|$ and $\dot{\vec{r}}$ is a derivative of $\vec{r}$ with respect to parameter t. Adding and subtracting the regularizer

$$\int_0^{2\pi} \frac{1}{2\sin^2\frac{t-t'}{2}} \xi(t')u(t')L(t')dt' \tag{5}$$

on the left-hand side of (4), we obtain the integral equation

$$\mathrm{FP} \int_0^{2\pi} \frac{v(t')}{4\pi \sin^2\frac{t-t'}{2}} dt' + \frac{1}{2\pi} \int_0^{2\pi} v(t')K(t,t')dt' = \widetilde{F}(t),\ \ t \in [0;2\pi], \tag{6}$$

with new unknown function $v(t') = 2\pi\xi(t')u(t')L(t')$ and new kernel

$$K(t,t') = \frac{1}{\xi(t')} \frac{\partial^2 G(\vec{r}(t),\vec{r}\,'(t'))}{\partial n \partial n'} - \frac{1}{2\sin^2\frac{t-t'}{2}}. \tag{7}$$

The auxiliary function $\xi(t)$ needs to be generated so that the kernel (7) would have a weak singularity only. For that, we take $\xi(t) = 1/4\pi L^2(t)$. With this $\xi(t)$, the function (7) has only an integrable singularity as $t \to t'$, viz.,

$$K(t,t') \sim -k^2 L^2(t') \ln \frac{k|\vec{r}(t) - \vec{r}'(t')|}{2} + \frac{k^2}{2} L^2(t') \left(1 - 2\gamma + \frac{\pi}{i} + \frac{\chi^2(t')}{k^2} \right), \tag{8}$$

where $\gamma = 0.5772...$ is the Euler constant, and $\chi(t') = \dot{\vec{r}}(t') \cdot \vec{n} / L^2(t')$ is the curvature of ∂D. Let seek a solution to (6) in $L_2(\Omega)$ as a Fourier expansion on a complete orthogonal set $\left\{ e^{iq't'} \right\}_{q'=-\infty}^{+\infty}$:

$$v(t') = \sum_{q'=-\infty}^{+\infty} c_{q'} e^{iq't'} . \tag{9}$$

Using this representation of unknown function, the formula

$$\mathrm{FP} \int_0^{2\pi} \frac{e^{iq't'}}{2\sin^2 \frac{t-t'}{2}} dt' = -2\pi |q'| e^{iq't}, \; q' = \{0; \pm 1; \pm 2; ...\}, \tag{10}$$

which is proved in the Appendix, and supposing the absolute convergence of the series (9) that permits us to change the integration and summation order, we obtain the equation

$$-\sum_{q'=-\infty}^{+\infty} c_{q'} |q'| e^{iq't} + \frac{1}{2\pi} \sum_{q'=-\infty}^{+\infty} c_{q'} \int_0^{2\pi} K(t,t') e^{iq't'} dt' = \widetilde{F}(t), \quad t \in [0; 2\pi]. \tag{11}$$

Multiplying this one by $e^{-iqt}/2\pi$ and integrating over $[0;2\pi]$, we obtain after dividing by $-|q| \, (q \neq 0)$ the following 2nd kind equation

$$c + Ac = f, \tag{12}$$

for unknown vector c of Fourier coefficients. The matrix operator A corresponds to the matrix

$$a_{qq'} = -\frac{1}{|q| + \delta_{q0}} \left(\frac{1}{4\pi^2} \int_0^{2\pi} e^{-iqt} \int_0^{2\pi} e^{iq't'} K(t,t') dt' \right) - \delta_{q0} \delta_{q'0}, \tag{13}$$

with $\delta_{qq'}$ being the Kroneker delta. The RHS vector has the following coordinates:

$$f_q = -\frac{1}{|q| + \delta_{q0}} \frac{1}{2\pi} \int_0^{2\pi} e^{-iqt} \widetilde{F}(t) dt \,. \tag{14}$$

Let prove that obtained equation (12) is Fredholm. To this end, it is sufficient to prove that the operator A is compact in the Hilbert space of sequences (l_2) and $f \in l_2$. Really, the operator A is compact in l_2 if the matrix (13) is square summable. From general properties of Fourier integrals it follows that due to the integrability of function $K(t,t')$ the double integral in (13) decays in any case as $|qq'|^{-1}$ with indexes ($|q|$ and $|q'|$) growth. Consequently, $|a_{qq'}| \sim |q^2 q'|^{-1}$. Accounting additionally for that the matrix ($\delta_{q0}\delta_{q'0}$) has only one nonzero element, we can conclude that the matrix (13) is square summable and hence, the operator A is compact in l_2. From the same properties of Fourier integral, it follows that integral in (14) decays in any case as $|q|^{-1}$ with $|q|$ growth and then, $|f_q| \sim q^{-2}$. This proves the needed summability of vector f in l_2. All this proves the equation (12) is Fredholm. As has been proved by von Koch [5], an equation such (12) has a unique solution if abovementioned conditions fulfill and the determinant of the system is nonzero. Because of the spectrum of integral operator in (6) coincides with that of its discretized counterpart in (12) when using the Galerkin method [6], the determinant of the system (12) is nonzero in the nonresonance case when solution to the Neumann problem exists and is unique [1]. Hence, in nonresonance case there exist the unique solution to equation (12), which belongs to l_2 [6]. This one ensures the absolute convergence of series (9) and justifies the changing of integration and summation order when obtaining the equation (12). An approximate solution

$$v_N(t) = \sum_{|q| \le N} c_q e^{iqt} \tag{15}$$

of truncated equation

$$c_q + \sum_{|q'| \le N} a_{qq'} c_{q'} = f_q, \; |q| \le N \tag{16}$$

converges in this case to an exact solution of integral equation (6) in L2 if $N \to \infty$.

3. ACTUAL ERROR

One of the main points of the Neumann problem solving procedure in integral equation formulation is the estimation of an actual error of a solution with the aid of residual error computed in L_2. The latter stands for the boundary condition error and can be easily computed in practice. This problem was considered in [4] in the energy Sobolev space. We consider this problem in the familiar L_2 space. The completeness of basis functions used permits us to obtain a rigorous relation between actual and residual errors. To this end, let write down an actual error of the approximate solution to the equation (6) as a function

$$\varphi_N(t) = v(t) - v_N(t) \tag{17}$$

Function (17) satisfies the equation

$$\int_0^{2\pi} \frac{\varphi_N(t')}{4\pi \sin^2 \frac{t-t'}{2}} dt' + \frac{1}{2\pi} \int_0^{2\pi} \varphi_N(t') K(t,t') dt' = E(t), \ t \in \Omega \ , \tag{18}$$

that is the same as (6), but with another RHS, which is now the discrepancy of equation (6) for approximate solution (15):

$$E(t) = \widetilde{F}(t) - \mathrm{FP} \int_0^{2\pi} \frac{v_N(t')}{4\pi \sin^2 \frac{t-t'}{2}} dt' + \frac{1}{2\pi} \int_0^{2\pi} v_N(t') K(t,t') dt' . \tag{19}$$

Hence, we can use the abovementioned technique to obtain a convergent GMoM solution for error function (17) making use of its Fourier representation

$$\varphi_N(t) = \sum_{q \in Z} \alpha_n e^{iqt} \tag{20}$$

in L_2. As is clear from (12), the corresponding solution in l_2 can be written then as

$$\alpha = (I + A)^{-1} b , \tag{21}$$

where vector b has the coordinates

$$b_q = -\frac{1}{(|q| + \delta_{q0})} \frac{1}{2\pi} \int_0^{2\pi} e^{-iqt} E(t) dt \ . \tag{22}$$

Analogously to the above, the obtained solution (21) yields the convergent approximate solution (20) if the right-hand side vector b will be square summable. This will be so if the discrepancy function E(t) will be integrable over $[0,2\pi]$. Let show that it is so. The function $\widetilde{F}(t)$ is obviously integrable. Using (10) and (15), one can easily obtain

$$\mathrm{FP}\int_0^{2\pi} \frac{v_N(t')}{4\pi \sin^2 \frac{t-t'}{2}} dt' = \sum_{|q|\le N} c_q \mathrm{FP}\int_0^{2\pi} \frac{e^{iqt'}}{4\pi \sin^2 \frac{t-t'}{2}} dt' = -\sum_{q\le N} c_q \,|\, q \,|\, e^{iqt} \,. \tag{23}$$

The last expression shows that the first expression in (23) is continuous and hence integrable for any N. The last term in (19) is continuous and integrable because K(t,t') is integrable. All this proves the convergence of GMoM solution to the equation (18) and permits us to use the Parseval equality

$$\| \varphi_N(t) \|_{L_2} = \| \alpha \|_{l_2} \,, \tag{24}$$

which gives one the possibility to relate actual and residual errors in the form

$$\| v - v_N \|_{L_2} = \| (I + A)^{-1} b \|_{l_2} \,. \tag{25}$$

The actual error of the (2N+1)-term approximation $u_N(t)$ to the exact solution $u(t)$ of the initial solution (4) one can estimate then as follows:

$$\| u - u_N \|_{L_2} = 2 \| L(v - v_N) \|_{L_2} \le 2 \| L \|_{L_2} \| v - v_N \|_{L_2} \,. \tag{26}$$

4. COMPUTATIONAL FEATURES

The relative boundary error e_N is computed as the Euclidean norm of residual error for (2N+1)-terms approximation to an exact solution divided by $\| H_0 \|_2$ versus progressively greater value of the grid resolution $g_r = \lambda / h$ (a number of subintervals or points per wavelength) with h the grid spacing. The latter parameter, g_r, has been proposed in [7] instead of the number of subintervals (n) and has, to our opinion, an insightful physical meaning. Because of comparing the approximate solutions for scattering from circular cylinders with the exact Rayleigh solutions in the far zone, we compute the corresponding matching error as

$$e_R = \| F_R(\phi) - F(\phi) \|_2 / \| F_R(\phi) \|_2 \tag{27}$$

where $F_R(\phi)$ is the scattering amplitude function resulted from the Rayleigh solution and $F(\phi)$ is the corresponding scattering amplitude function resulted from the considered method. This error is computed on uniform subdividing of interval $[0,2\pi]$ by 2000 parts. When using the GMoM, the truncation number N is related with n as $N = [n/2]$, to support the needed accuracy when evaluating the peripheral MoM matrix elements. The number (n) of subintervals, which are used for approximate evaluation of the MoM matrix elements, is derived from the formula: $n = [\,|\partial D|\, kg_r / 2\pi]$, where $|\partial D|$ is the boundary length and $[\cdot]$ is the integer part of a number. To compare different numerical algorithms in terms of total computational labor (TCL), we estimate this notion as a numeric quantity. The TCL can be estimated as the ratio of CPU time, $t_{CPU}(ka,n)$, of tested algorithm enacted on given meshing (n) to that of any standard numerical one with the same meshing. As such a standard algorithm, we use the (2N+1)-terms approximation to the MFIE solution with the aid of GMoM for scattering of an H-polarized plane wave by a circular metallic cylinder of the same electric size (ka). The CPU time of this standard algorithm we denote as $t_{st}(ka,n)$. Then, the TCL can be estimated numerically as

$$tcl = t_{CPU}(ka,n) / t_{st}(ka,n)\,. \tag{28}$$

Such an estimation of the main characteristic of an algorithm, permits one to compare different numerical methods independently from their machinery realization. Besides this characteristic, we also use a monostatic RCS (MRCS) in unitless form

$$\sigma_b = \lim_{\rho\to\infty} (2\pi k\rho\, |H_{sc}|^2 / |H_0|^2)\,. \tag{29}$$

The Fourier coefficients (c_q) for approximate solution (15) were derived from (16) with matrix elements $a_{qq'}$ evaluated in the Bogolyubov-Krylov approach [8] as

$$a_{qq'} = \frac{2\pi h^2}{|q| + \delta_{q0}} \sum_{k,s=1}^{n} e^{-iqt_k} D_{ks} e^{-iq't_s}\,. \tag{30}$$

where

$$D_{ks} = \begin{cases} K(t_k, t_s), & t_k \neq t_s \\ \widetilde{K}(t_s), & t_k = t_s \end{cases}$$

and

$$\widetilde{K}(t_s) = \frac{1}{h}\int_{t_s-d}^{t_s+d} K(t_s, t')dt' =$$

$$\frac{1}{h}\int_{t_s-d}^{t_s+d}\left(-k^2L^2(t')\ln\frac{k}{2}\left|t_s - t'\right| + K(t_s, t') + k^2L^2(t')\ln\frac{k}{2}\left|t_s - t'\right|\right)dt' \approx$$

$$-\frac{k^2L^2(t_s^0)}{h}\int_{t_s^0-d}^{t_s^0+d}\ln\left|t_s^0 - t'\right|dt' + -k^2L^2(t_s^0)\ln L(t_s^0) + \frac{k^2}{2}L^2(t_s^0)\left(1 - 2\gamma + \frac{\pi}{i} - 2\ln\frac{k}{2} + \frac{\chi^2(t_s^0)}{k^2}\right)$$

with $d = h/2$. An improvement of this approach is possible, in principle, but leads to significant increasing in computational labor and in fact is not effective. At last, the upper estimation of relative actual error was computed as

$$e_N^{(\psi)} = \| L \|_{L_2} \| v - v_N \|_{L_2} / \| Lv_N \|_{L_2} \tag{31}$$

5. NUMERICAL ANALYSIS

Because of a rigorous solution to the HSIE with GMoM has not been obtained before now, as to authors' knowledge, we provide in this work a detailed comparison of obtained HSIE solutions with that of the MFIE for the Neumann problem. The main scattering characteristics for circular cylinder of ka = 100 obtained with the new method solution of the HSIE are collected in Table 1. They show a sufficiently high accuracy of the new method ($e_N, e_N^{(\psi)} \sim 10^{-4}$) with the same computational cost as for MFIE solutions (tcl ~ 1). The comparison of scattering amplitude functions for HSIE and Rayleigh solutions by evaluating the matching error (27) confirms the high accuracy and adequacy of used algorithm. Both actual and residual errors are decreasing with g_r growing, as is also the condition number of the matrix equation. The corresponding characteristics obtained by solving the MFIE are listed in Table 2. As can be supposed, the solutions to the MFIE are more accurate than that of to the HSIE. The

both tables show the tendency of MRCS (σ_b) to the exact value ($\sigma_R = 24.9712$ dB) obtained with the aid of high accurate Rayleigh solution (with relative error ~ 10^{-14}).

Table 1. HSIE. Circular cylinder, $ka = 100$, $i_0 = 45^\circ$

$g_r\,(n)$	5 (500)	6 (600)	7 (700)	8 (800)	9 (900)	10 (1000)	11 (1100)
N	250	300	350	400	450	500	550
$e_N \cdot 10^4$	0.26	0.21	0.18	0.16	0.14	0.12	0.10
$e_N^{(\psi)} \cdot 10^4$	2.1	1.2	1.5	1.4	1.2	1.0	0.9
$e_R \cdot 10^3$	5.4	5.6	5.3	4.9	4.5	4.1	3.8
σ_b, dB	24.9332	24.9280	24.9330	24.9395	24.9453	24.9499	24.9535
$cond \cdot 10^{-3}$	7.1	7.2	6.9	6.6	6.4	6.2	6.0
tcl	1.11	0.93	0.91	0.88	0.83	0.85	0.87

Table 2. MFIE. Circular cylinder, $ka = 100$, $i_0 = 45^\circ$

$g_r\,(n)$	5 (500)	6 (600)	7 (700)	8 (800)	9 (900)	10 (1000)	11 (1100)
$e_N \cdot 10^7$	5.52	2.41	1.23	0.69	0.42	0.27	0.18
$e_N^{(\psi)} \cdot 10^6$	7.74	3.45	1.78	1.02	0.62	0.40	0.27
$e_R \cdot 10^3$	1.31	0.72	0.44	0.29	0.20	0.15	0.11
σ_b, dB	24.9668	24.9683	24.9693	24.9699	24.9703	24.9705	24.9707
$cond$	183.8	183.8	183.8	183.8	183.8	183.8	183.8

With diminishing of thickness of a scatterer both HSIE and MFIE solutions become less accurate. Scattering characteristics obtained with these equations for not very elongated elliptic cylinder of ka = 100 and b/a = 0.1 are listed in Table 3 and 4 consequently. We can see in these tables some diminishing in accuracy for both methods.

Table 3. HSIE. Elongated cylinder, $ka = 100$, $i_0 = 45^\circ$, $b/a = 0.1$.

$g_r\,(n)$	5 (323)	6 (388)	7 (453)	8 (517)	9 (582)	10 (647)	11 (711)
N	162	194	227	259	291	324	356
$e_N \cdot 10^4$	0.97	0.35	0.24	0.22	0.20	0.18	0.16
$e_N^{(\psi)} \cdot 10^4$	2.7	1.2	1.1	1.1	1.0	0.8	0.8
σ_b, dB	8.3102	8.3139	8.3131	8.3098	8.3060	8.3023	8.2991
$cond \cdot 10^{-4}$	5.9	1.5	0.8	0.8	0.8	0.8	0.8
tcl	1.30	1.33	1.18	1.13	1.13	1.07	1.05

Table 4. MFIE. Elongated cylinder, *ka* = 100, $i_0 = 45^\circ$, *b/a* = 0.1.

g_r (n)	5 (323)	6 (388)	7 (453)	8 (517)	9 (582)	10 (647)	11 (711)
$e_N \cdot 10^7$	3.65	1.48	0.73	0.41	0.25	0.16	0.11
$e_N^{(\psi)} \cdot 10^7$	18.8	8.4	4.3	2.5	1.5	1.0	0.7
σ_b, dB	8.2669	8.2677	8.2681	8.2682	8.2683	8.2684	8.2684
cond	64.5	64.5	64.5	64.5	64.5	64.5	64.5
tcl	1.02	1.12	1.12	1.10	1.11	1.11	1.11

Numerical investigation shows small diminishing in accuracy for b/a being in the range from 1 to 0.1, but for b/a < 0.1 the accuracy decreases very rapidly with b/a. A MFIE solution may be incorrect at all for b/a less than 0.05. A HSIE solution remains correct in any case for $b/a \geq 0.01$. However, the accuracy of a solution for such a thin scatterer is not very high. The scattering characteristics for very elongated elliptic cylinder with b/a = 0.01 are collected in Table 5. Note that condition number increases

Table5. HSIE. Elongated cylinder, *ka* = 100, $i_0 = 45^\circ$, *b/a* = 0.01.

g_r (n)	6 (382)	7 (446)	8 (255)	9 (573)	10 (637)	11 (700)	12 (764)	13 (828)	14 (892)
N	191	223	255	287	319	350	382	414	446
$e_N \cdot 10^3$	185.0	183.1	258.8	102.7	67.9	50.4	39.5	31.4	24.9
$e_N^{(\psi)} \cdot 10^3$	50.6	68.7	130.6	29.6	12.9	5.7	3.6	2.4	1.8
σ_b, dB	-2.9451	-2.0331	-1.7699	-1.7918	-1.8340	-1.8770	-1.9151	-1.9453	-1.9678
cond	$1 \cdot 10^5$	$6 \cdot 10^5$	$1 \cdot 10^6$	$8 \cdot 10^5$	$6 \cdot 10^5$	$3 \cdot 10^5$	$3 \cdot 10^5$	$3 \cdot 10^5$	$3 \cdot 10^5$
tcl	1.32	1.20	1.17	1.18	1.10	1.08	1.03	0.96	1.01

with the thickness of a scatterer diminishing and for b/a = 0.01 it is about of 10^5. The increasing of condition number observed in Table 5 for small g_r is caused by the lack of accuracy when evaluating matrix elements. For larger g_r the condition number becomes less and decreases to some level. In that range of g_r, where the condition number increases, a solution may be inadequate and we have to enlarge g_r to obtain an adequate solution. In general, the accuracy of GMoM solution to the HSIE increases with ka growth for given g_r as is shown in Table 6 for g_r = 10, ka = 100, and $i_0 = 45^\circ$. It should be interesting to demonstrate a difference in scattering amplitude

functions obtained by solving the HSIE and MFIE but they are completely overlapping for all $g_r \geq 5$ as one can conclude from Tables 1-4. Because of that, we show in Figure 1 the scattering amplitude functions for elongated cylinder of ka = 100 and $i_0 = 45^\circ$ obtained by solving the HSIE only for different axial ratios (b/a = 0.1, 0.05, 0.01). Here we can see the scattering amplitude functions approaching some hypothetical symmetric pattern of a strip. In Figure 2, a magnified part of the central area of Figure 1 is shown, in which we can see specific features of scattering from elongated cylinder. We can see herein the substantial changing of backward scattering with b/a diminishing and practically unchanged forward scattering.

Table 6. HSIE. Elliptic cylinder of *ka* = 100, *b/a* = 0.1 ($i_0 = 45^\circ$).

ka (*n*)	10 (65)	30 (194)	50 (323)	70 (453)	90 (582)	110 (711)
$e_N^{(\psi)} \cdot 10^4$	98.1	2.7	4.5	1.0	0.7	0.3
σ_b , dB	2.5417	4.2498	6.2102	7.3491	8.2985	9.2206
cond	557	6220	3390	4710	9820	8800

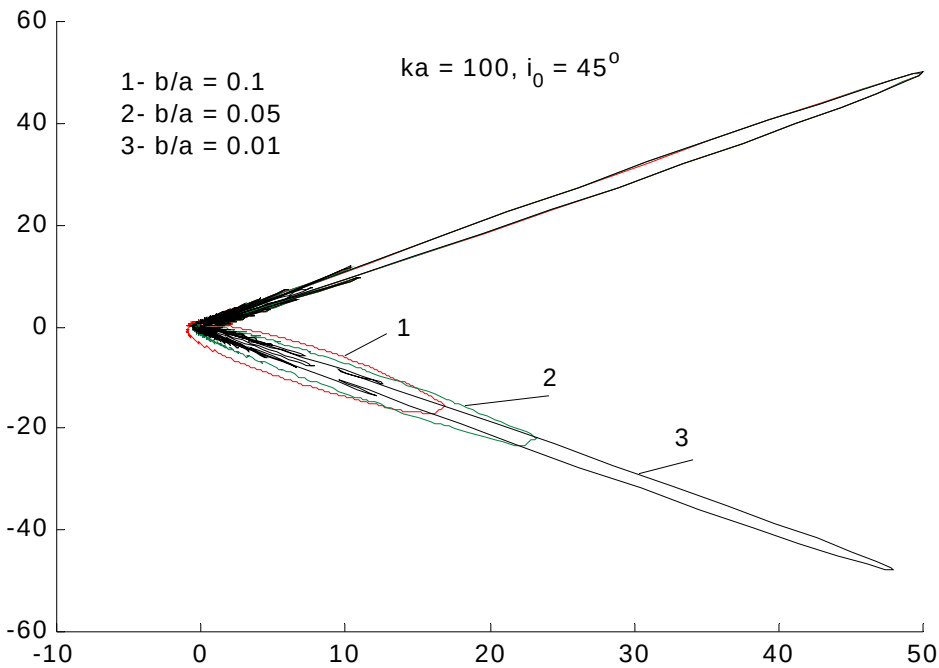

Figure 1. Scattering amplitude functions for elongated cylinders of *ka* = 100 and oblique incidence ($i_0 = 45^\circ$) for different axial ratios (b/a = 0.1, 0.05, 0.01).

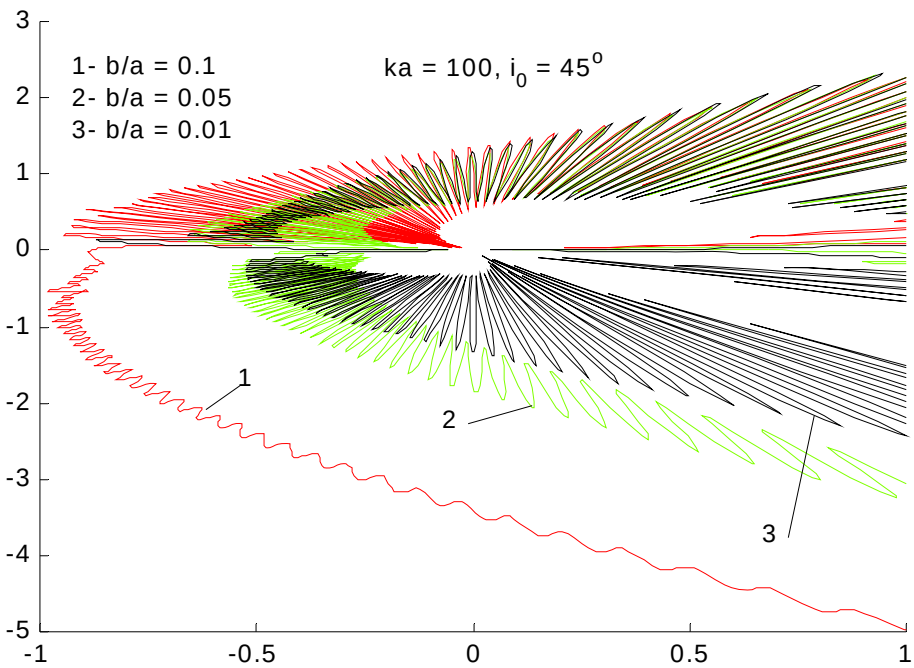

Figure 2. The magnified part of the central area of Figure 1.

6. CONCLUSIONS

In the present work, the convergence of the Galerkin MoM has been proved in L_2 for the hypersingular 1st kind integral equation of Neumann problem for two-dimensional object with a smooth boundary in the nonresonance case. Practically convenient Fredholm 2nd kind equation in the space of sequences has also been obtained for Fourier coefficients of the HSIE solution. To this end, the method of analytical regularization has been used with exact evaluation of hypersingular integrals over basis functions as the finite part in the sense of Hadamard. The proved convergence of GMoM on a complete set of basis functions permitted us to obtain an exact relation between actual and residual errors in L_2. This, in turn, permits one to calculate an actual error simply by integration of residual error along the boundary.

The Neumann problem solutions derived with proposed method have been compared both with exact Rayleigh solutions for circular cylinder and GMoM solutions to the MFIE for circular and elongated cylinders. The Neumann problem solution for very elongated elliptic cylinder has been also considered. The obtained results are also

applicable for solving the combined field integral equation [1]. These one will be considered in the next work.

APPENDIX

Let $u(t)$ is a 2π-periodic function, which derivative is Lipshitz, and there exist the following integral

$$\mathrm{FP}\int_0^{2\pi}\frac{\mathrm{u(t')}}{2\sin^2\frac{t'-t}{2}}dt'=\lim_{\varepsilon\to+0}\left\{\int_0^{t-\varepsilon}\frac{\mathrm{u(t')}}{2\sin^2\frac{t'-t}{2}}dt'+\int_{t+\varepsilon}^{2\pi}\frac{\mathrm{u(t')}}{2\sin^2\frac{t'-t}{2}}dt'-\frac{4u(t)}{\varepsilon}\right\} \tag{A1}$$

as a finite part in the sense of Hadamard. Making use of integrating by parts in

$$\int_0^{t-\varepsilon}\cot\frac{t'-t}{2}\dot{u}(t')dt'+\int_{t+\varepsilon}^{2\pi}\cot\frac{t'-t}{2}\dot{u}(t')dt',$$

we obtain

$$\int_0^{t-\varepsilon}\frac{u(t')}{2\sin\frac{t'-t}{2}}dt'+\int_{t+\varepsilon}^{2\pi}\frac{u(t')}{2\sin\frac{t'-t}{2}}dt'-(u(t-\varepsilon)+u(t+\varepsilon))\cot\frac{\varepsilon}{2}.$$

Then, using (A1) we have at the limit $\varepsilon\to+0$

$$\mathrm{PV}\int_0^{2\pi}\cot\frac{t'-t}{2}\dot{u}(t')dt'=\mathrm{FP}\int_0^{2\pi}\frac{\mathrm{u(t')}}{2\sin\frac{\mathrm{t'-t}}{2}}dt'. \tag{A2}$$

With the use of known Hilbert transform of $u(t)=\exp(iqt)$, viz.,

$$\mathrm{PV}\int_0^{2\pi}\cot\frac{t'-t}{2}\exp(iqt')dt'=\begin{cases}2\pi i\frac{q}{|q|}\exp(iqt), & |q|>0\\ 0, & q=0\end{cases},$$

we can easily obtain from (A2) the needed formula:

$$\mathrm{FP}\int_0^{2\pi}\frac{\mathrm{u(t')}}{2\sin\frac{\mathrm{t'-t}}{2}}dt'=\begin{cases}-2\pi i|q|\exp(iqt), & |q|>0\\ 0, & q=0\end{cases}. \tag{A3}$$

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