# ORTHOSYNTHESIS: ON THE BEAMFORMING FROM A SET OF UNIFORM LINEAR ARRAYS BY THE ORTHOGONAL METHOD

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## Abstract

*A method of beamforming of uniform linear arrays by using the orthogonalization technique is presented. A set of composing functions similar to that used by Woodward – Lawson is overlapped. In our study, instead of sampling of the desired pattern, the orthogonal method is applied. Depending on the form of the pattern, the number of the composing functions may be the same or different than the elements of the array. Also the progressive phase of each function can be derived in several ways. A set of numerical examples for different array patterns will be presented and will show the usefulness of the method.*

## 1. INTRODUCTION

Synthesis methods for linear antenna arrays have been studied by many researchers in the last decades; the existence of a long series of papers on this subject is enough to emphasize the importance of the area. Most of the procedures allow the synthesis of narrow-beam or low sidelobe patterns or the maximization of an index (gain, SNR) subject to one or several constraints. Excellent textbooks in the international literature, [1-4], present several treatments on the synthesis problem. Among the above treatments we notice the Fourier transform method [2], the Schelkunoff procedure [5], the Dolph-Chebyshev and the Riblet synthesis [6, 7], the the Woodward - Lawson method [8-10] and the Orchard et al synthesis [11]. The orthogonal method was introduced by Unz [12] and has been extensively used by Sahalos [13] in many antenna synthesis problems. Our effort in this paper is the application of the above method for multiple beam beam-forming.

The procedure starts from a set of uniform, non-orthogonal progressively phased composing functions [9], which are orthonormalized and overlapped. The orthonormalized functions are weighted to form the desired pattern. Our method is general and able to interpolate between beams and to combine them in order to synthesize the desired one.

## 2. FORMULATION

Let a linear array (Figure 1) be composed of N equidistant point sources. The array factor is given by:

$$F_r(\theta) = \sum_{k=1}^{N} A_k e^{-j[N-(2k-1)]\frac{\beta d}{2}\cos\theta} \tag{1}$$

where $A_k$ is the excitation coefficient of the k$^{th}$ element, $\beta = \frac{2\pi}{\lambda}$ and d is the equidistance of the array elements.

The functional basis of (1) is:

$$\left\{ \Phi_k^o(\theta) = e^{-j[N-(2k-1)]\frac{\beta d}{2}\cos\theta} \right\} \tag{2}$$

It would be tiresome, if not tedious, to repeat here the procedures of orthonormalizing this basis and arriving at the formulas that give the complex amplitudes $A_k$. One can refer to [13] for the intermediate steps.

Figure 1. Geometry of an N element linear array

In our present work we suppose that the array factor comes from a set of $M$ different patterns of uniformly illuminated N element arrays. In this case $F_r(\theta)$ can be expressed in the following form:

$$F_r(\theta) = \sum_{m=1}^{M} I_m \Phi_m(\theta) \tag{3}$$

where

$$\Phi_m(\theta) = \frac{1}{N}\sum_{n=1}^{N} e^{-j[N-(2n-1)]\left(\frac{\beta d}{2}\cos\theta - \alpha_m\right)} \tag{4}$$

The functional base of (3) $\Phi_m(\theta)$ is not orthogonal and, in this case, the inner product $\langle \Phi_i, \Phi_j \rangle$ is given by:

$$\langle \Phi_i, \Phi_k \rangle = \frac{1}{N^2}\sum_{n=1}^{N}\sum_{l=1}^{N} S_{\ln} e^{j\{[N-(2l-1)]\alpha_i - [N-(2n-1)]\alpha_k\}} \neq 0 \tag{5a}$$

where

$$S_{\ln} = 2\pi \frac{\sin((l-n)\beta d)}{(l-n)\beta d} \tag{5b}$$

It is noticed that in the inner products there is a transformation of the independent variable $\theta$ to $u=\pi cos(\theta-\pi/2)=\pi sin(\theta)$.

It should be noted that the inner product is simplified if the angle $\theta$ is measured from broadside instead of the array axis [13]. In that case:

With the aid of (5), it is possible to construct $M$ new functions, related to each other by a relation analogous to that of orthogonality, following the Gram-Schmidt procedure [13].

We thus get:

$$Y_k = \Phi_k - \sum_{i=1}^{k-1} \frac{\langle Y_i, \Phi_k \rangle}{\langle Y_i, Y_i \rangle} Y_i \tag{6}$$

Expression (6) is divided by $\left|\langle Y_k, Y_k \rangle\right|^{1/2}$ so that the resulting function will be orthonormalized. It is:

$$\Psi_k = \frac{Y_k}{\left|\langle Y_k, Y_k \rangle\right|^{1/2}} \tag{7}$$

where $\Psi_k$ and $\Psi_j$ will be related by:

$$\langle \Psi_k, \Psi_j \rangle = \delta_{kj} \tag{8}$$

$F_r(\theta)$ can now be expressed as a function of $\Psi_i(\theta)$ as follows:

$$F_r(\theta) = \sum_{i=1}^{M} B_i \Psi_i(\theta) \tag{9}$$

If we suppose that a pattern $F_d(\theta)$ is desired and $F_r(\theta)$ can approximate it, then in view of (8) $B_i$ will be given by:

$$B_i = \langle F_d(\theta), \Psi_i(\theta) \rangle \tag{10}$$

The quantities $I_i$ in view of (9) are:

$$I_i = \sum_{j=i}^{M} B_i C_i^{(j)} \tag{11}$$

It is noticed that expression (10) comes from the minimization of the mean square error (MSE). It is:

$$MSE = \left\langle \left\| F_r(\theta) - F_d(\theta) \right\|^2 \right\rangle = \min \tag{12}$$

Eq. (12) gives:

$$\frac{\partial(MSE)}{\partial B_i} = 0 \text{, for } i=1,2,\ldots,M \tag{13}$$

Obviously, (10) can be derived from expression (13).

From (6) – (8) one can get the coefficients $C_i^{(j)}$ in the following form:

$$C_n^{(n)} = \frac{1}{\left| \langle Y_n, Y_n \rangle \right|^{1/2}} \tag{14}$$

$$C_k^{(n)} = -C_n^{(n)} \sum_{j=k}^{n-1} C_k^{(j)} \sum_{i=1}^{j} C_i^{(j)} \langle \Phi_i, \Phi_j \rangle \tag{15}$$

Taking into account expressions (1)-(3) the array element excitation are derived as:

$$A_k = \sum_{m=1}^{M} \lambda_{mk} I_m \tag{16}$$

where

$$\lambda_{mk} = e^{j[N-(2k-1)]\alpha_m} \tag{17}$$

$\alpha_m$ of (4) and (17) could be defined in several ways. A suitable form could be:

$$\alpha_m = \alpha \frac{[M - (2m-1)]\pi}{2N} \tag{18}$$

For $\alpha = 1$ the phase of the m$^{th}$ composing function $\Phi_m(\theta)$ makes the main lobe maximum coincident with the innermost null of the (m-1)$^{th}$ corresponding one. In this case, where *M=N*, the composing functions are the same with those used in the Woodward – Lawson method. It must be pointed out that the above method is based on the sampling concept of the desired pattern and its defect is the lack of control over the sidelobe level in the unshaped region of the pattern. Our method makes use of

composing functions of more general type and, instead of sampling, it applies the orthonormalization procedure. Taking different values of $\alpha$ and $M$ in (18) we receive different solutions of the synthesis problem. It is also possible to specify certain values of $\alpha_m$ which could improve the solution. In the examples section several cases with different choices of $M$ and $N$ will be presented.

The synthesis procedure can be summarized in the following steps:

1. Definition of the number $N$ and the equidistance $d$ of the array elements.
2. Definition of the number $M$ of the composing functions and their corresponding linear phase $a_m$.
3. Calculation of coefficients $C_i^{(j)}$.
4. Evaluation of the desired pattern $F_d(\theta)$.
5. Calculation of the quantities $B_i$ and $I_i$.
6. Calculation of the excitations $A_i$ of the array elements.

Our method has no restrictions on the element numbers and the pattern symmetry. Also the degrees of freedom, in contrast to Woodward-Lawson are the maximum possible.

## 3. EXAMPLES

As a first group of examples we will apply the method to the design of Chebyshev patterns. The design cases correspond to the ones presented in [14]. It is supposed that an endfire $T_5(x)$ pattern with SLL=-20dB is desired. For the 1st case, where the progressive phase shift is $\alpha = -\beta d$, using $M$=$N$=11 the excitation coefficients obtained by our method are exactly the same with these obtained by the Riblet analytical method for an array with equidistance $d$=0.25λ. Decreasing $M$ and keeping $N$=11 it is found that for up to $M$=8 different solutions give similar results. It must be pointed out that for $M$=$N$=8 the desired pattern can be produced. The excitation coefficients obtained in each of the cases mentioned above can be seen in Table 1, while Figure 2 represents the corresponding patterns. It is noticed that instead of 11 the problem can be solved with as few as 8 elements. Of course the MSE in the case of 11 elements is 100 times less than the corresponding one for N=8.

Next the method is applied to an array corresponding to the 2nd case, where the progressive phase shift is $\alpha = \beta d$ . Using $N$=9 elements, instead of the eleven that the Riblet method requires, the pattern obtained has exactly the same form. Table 2 and Figure 3 give the amplitudes and the corresponding pattern of the array for a $T_5(x)$ polynomial and $d = 0.2\lambda$ .

Table 1. Excitation coefficients of arrays with $d$=0.25$\lambda$, for the $T_5(x)$ Chebyshev polynomial corresponding to the 1st Riblet endfire case.

| Element Number | Excitation coefficients | | | | | |
|---|---|---|---|---|---|---|
| | $M$=$N$=11 | | $M$=8, $N$=11 | | $M$=$N$=8 | |
| | Magnitude | Phase (deg) | Magnitude | Phase (deg) | Magnitude | Phase (deg) |
| 1 | 0.671 | 90 | 0.591 | 86.5 | 0.13 | 165 |
| 2 | 0.567 | 0 | 0.405 | -0.304 | 0.429 | -6.50 |
| 3 | 0.735 | -90 | 0.651 | -79.8 | 0.781 | 179 |
| 4 | 0.874 | 180 | 0.818 | -179 | 1 | 0 |
| 5 | 0.967 | 90 | 0.856 | 94.3 | 1 | 179 |
| 6 | 1 | 0 | 1 | 0 | 0.781 | 0.575 |
| 7 | 0.967 | -90 | 0.856 | -94.3 | 0.429 | -174 |
| 8 | 0.874 | -180 | 0.818 | 179 | 0.13 | 14.0 |
| 9 | 0.735 | 90 | 0.651 | 79.8 | | |
| 10 | 0.567 | 0 | 0.405 | 0.304 | | |
| 11 | 0.671 | -90 | 0.591 | -86.5 | | |

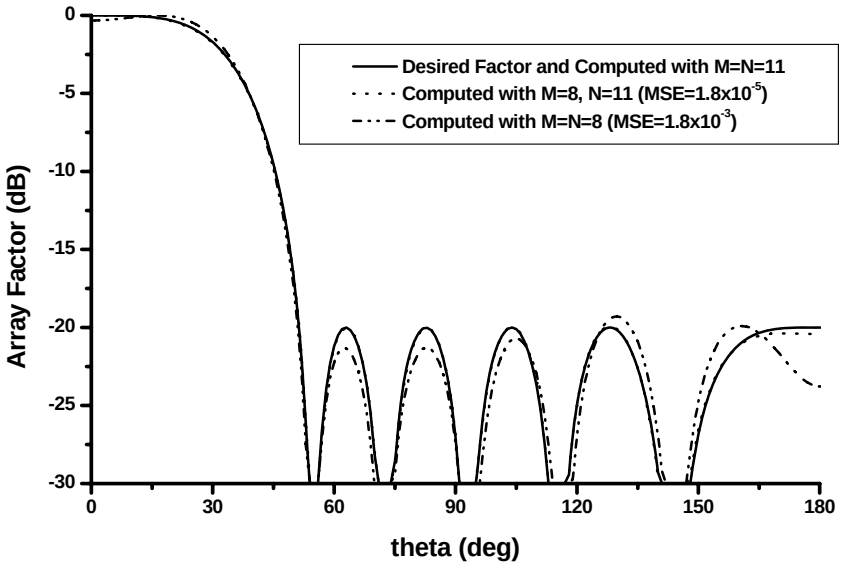



Figure 2. Computed Array Factor of arrays with $d$=0.25$\lambda$, for the $T_5(x)$ Chebyshev polynomial corresponding to the 1st Riblet endfire case.

Table 2. Excitation coefficients of an array with $d$=0.2$\lambda$, for the $T_5(x)$ Chebyshev polynomial corresponding to the 2nd Riblet endfire case.

| Element Number | Excitation coefficients | |
|---|---|---|
| | $M$=11, $N$=9 | |
| | Magnitude | Phase (deg) |
| 1 | 0.077 | 1.56 |
| 2 | 0.274 | -178 |
| 3 | 0.566 | -0.255 |
| 4 | 0.865 | 178 |
| 5 | 1 | 0 |
| 6 | 0.865 | -178 |
| 7 | 0.566 | 0.255 |
| 8 | 0.274 | 178 |
| 9 | 0.077 | -1.56 |

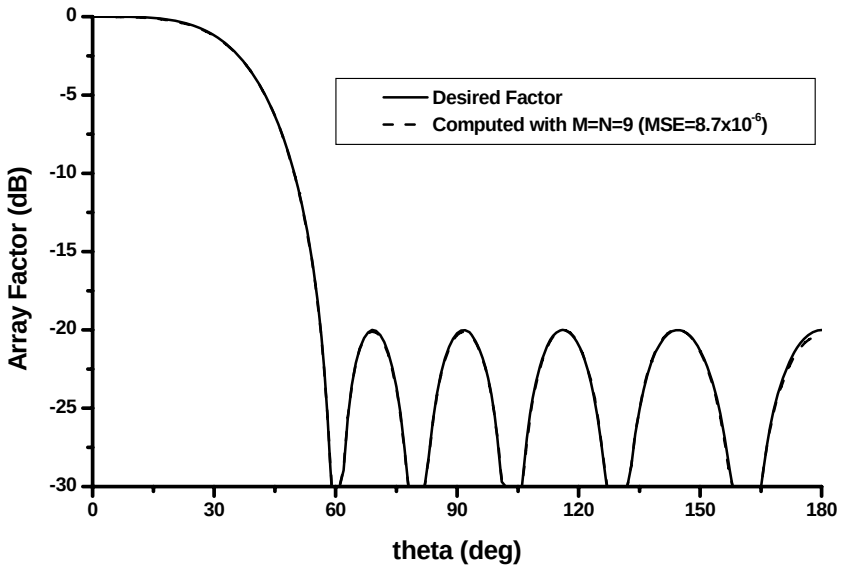



Figure 3. Computed Array Factor of an array with $d$=0.2$\lambda$, for the $T_5(x)$ Chebyshev polynomial corresponding to the 2nd Riblet endfire case.

In our next example we apply the method to the design of an optimum endfire array, which corresponds to the 3rd case of [14]. In this case the progressive phase shift is greater than the necessary for the classical endfire design. For $M$=$N$=11 the same results with the analytical method are found. Finally, for the 4th case we ask for a $T_5(x)$ array with SLL=-20dB and HPBW=30$^o$. Table 3 and Figure 4 show the excitation coefficients and the radiation patterns for $M$=$N$=11 and $M$=$N$=9 elements,

which give similar solutions with acceptable MSEs. In both cases the equidistance of the array is $d = 0.2\lambda$.

Table 3: Excitation coefficients of arrays with $d$=0.2$\lambda$, for the $T_5(x)$ Chebyshev polynomial corresponding to the 4th Riblet endfire case.

| Element Number | Excitation coefficients | | | |
|---|---|---|---|---|
| | $M$=$N$=11 | | $M$=$N$=9 | |
| | Magnitude | Phase (deg) | Magnitude | Phase (deg) |
| 1 | 0.002 | 152 | 0.017 | -22.7 |
| 2 | 0.027 | -22.3 | 0.131 | 163 |
| 3 | 0.152 | 163 | 0.432 | -11.3 |
| 4 | 0.453 | -11.2 | 0.816 | 174 |
| 5 | 0.824 | 174 | 1 | 0 |
| 6 | 1 | 0 | 0.816 | -174 |
| 7 | 0.824 | -174 | 0.432 | 11.3 |
| 8 | 0.453 | 11.2 | 0.131 | -163 |
| 9 | 0.152 | -163 | 0.017 | 22.7 |
| 10 | 0.027 | 22.3 | | |
| 11 | 0.002 | -152 | | |

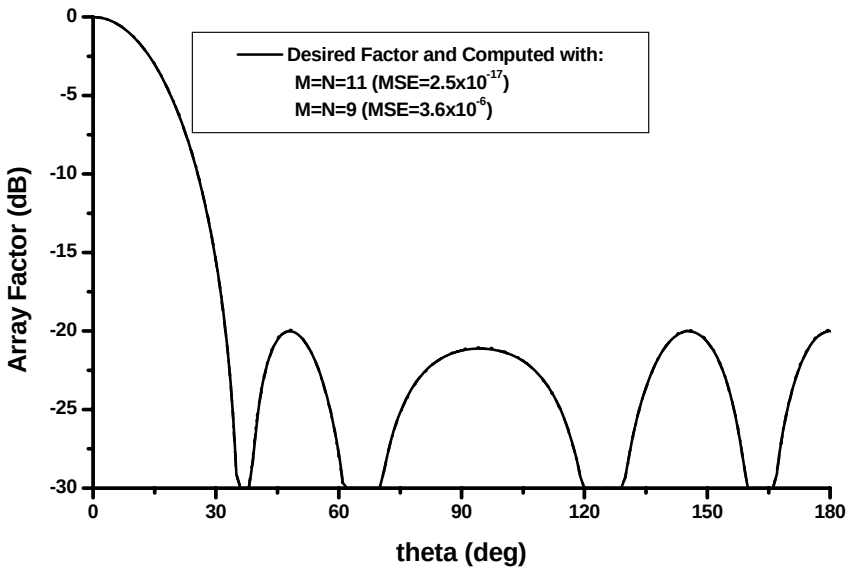
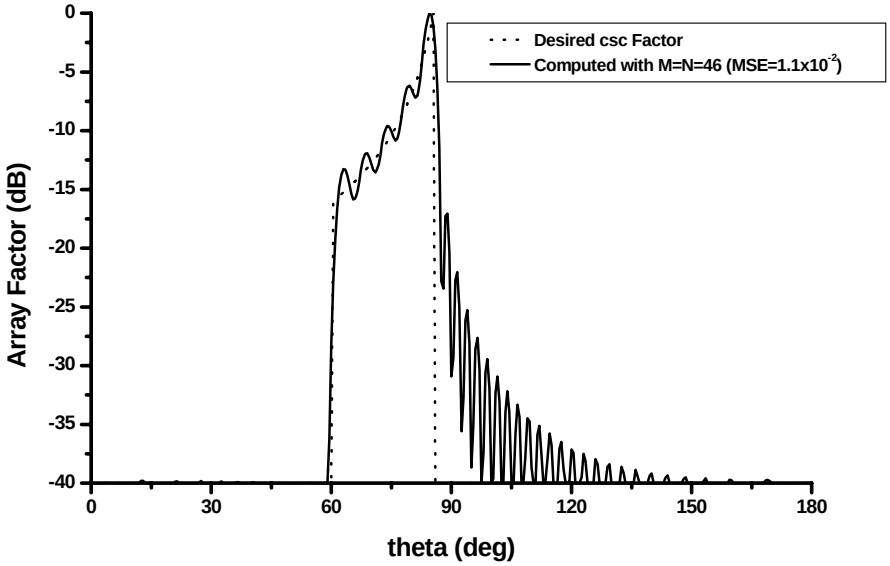
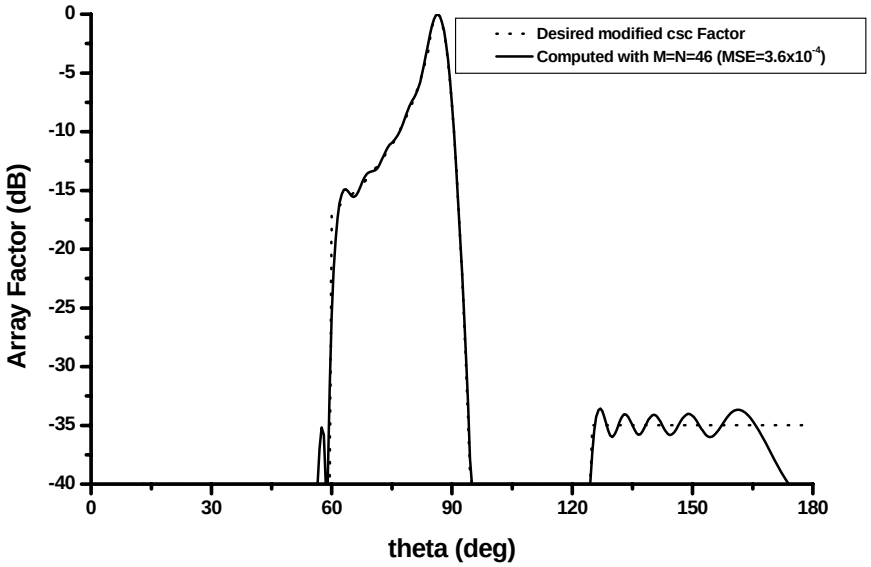



Figure 4. Computed Array Factor of arrays with $d$=0.2$\lambda$, for the $T_5(x)$ Chebyshev polynomial corresponding to the 4th Riblet endfire case.

The above examples show with convincingness the ability of the method to solve the Chebyshev synthesis problems with less (odd or even) than the required by the classical methods elements.




(a)




(b)

Figure 5. Desired and Computed Array Factor of an array with $d$=0.5$\lambda$, corresponding to (a) the cosecant function, and (b) the modified cosecant function.

Another interesting case is the design on an antenna with a cosecant – squared pattern. It is supposed that the beam is between $\theta = 60^o$ and $\theta = 85.4^o$. The array consists of $N$=46 elements with an equidistance of $d = 0.5\lambda$. Figure 5a shows the

pattern of our design for $M$=$N$=46. It is noted that the resulting pattern has a fair amount of ripple in it and the relative amplitude of the currents is 17.07dB. If in the vicinity of the maximum the pattern is expressed in an exponential form of the type $\exp\left[k(\theta-\theta_o/2)^2\right]$ , where $k$ is derived to have a HPBW equal to $\theta_o$, the resulting pattern becomes considerably smooth. It can be noticed that a sidelobe envelope can follow the requirements. Figure 5b shows the pattern for $M$=$N$=46 and $\theta_o = 4.6^o$. The desired pattern requires a –35dB level from 125$^o$ to 180$^o$. A disadvantage of the above array is that the relative amplitude of the currents becomes about 35.33dB. We could overcome this disadvantage by decreasing the HPBW of the desired pattern. As the desired HPBW decreases so the resultant pattern increases its ripple and its sidelobe level. The above example which contains pattern discontinuities shows the ability of the method to control the sidelobes.

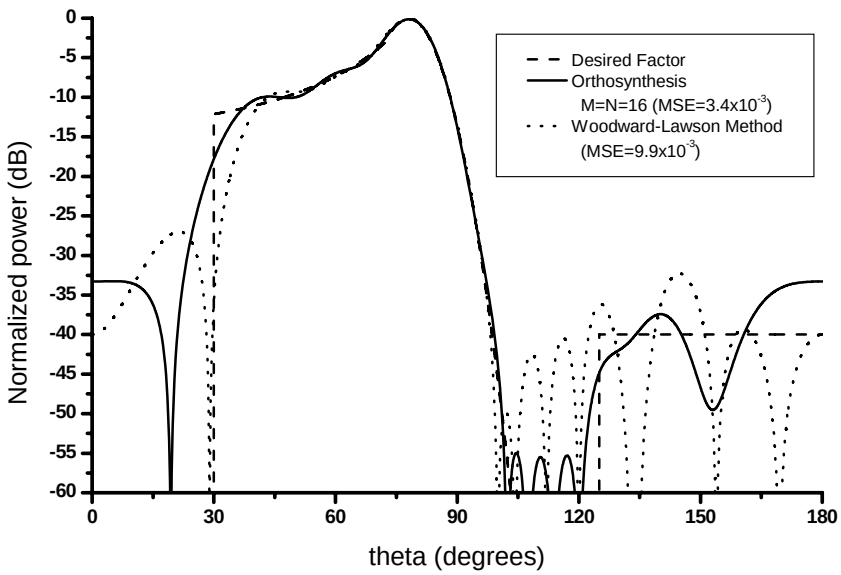



Figure 6 : Desired and Computed Array Factor of an array with $d$=0.5$\lambda$, corresponding to the modified cosecant function.

The same pattern with more requirements on the sidelobe envelope is presented. This case has to do with the control of the level of the close sidelobes as in the case of the Elliot's method. Sixteen elements with half-wavelength distance are used. The close sidelobes are controlled from the value of the HPBW of the pattern. The beam is between 30$^o$ and 78$^o$. Also the SLL envelope from 125$^o$ to 180$^o$ is ≤-30dB. Up to 125$^o$ there are sidelobes with level depended on the HPBW. A HPBW of 12.5$^o$ shows the close sidelobe level –55dB (see Figure 6). Obviously the example gives equivalently

acceptable results with these given by Elliot et al (see [15]). Comparing the result with the one obtained by the Woodward-Lawson method one can see that our technique offers almost 3 times more accurate results in terms of the mean square error.

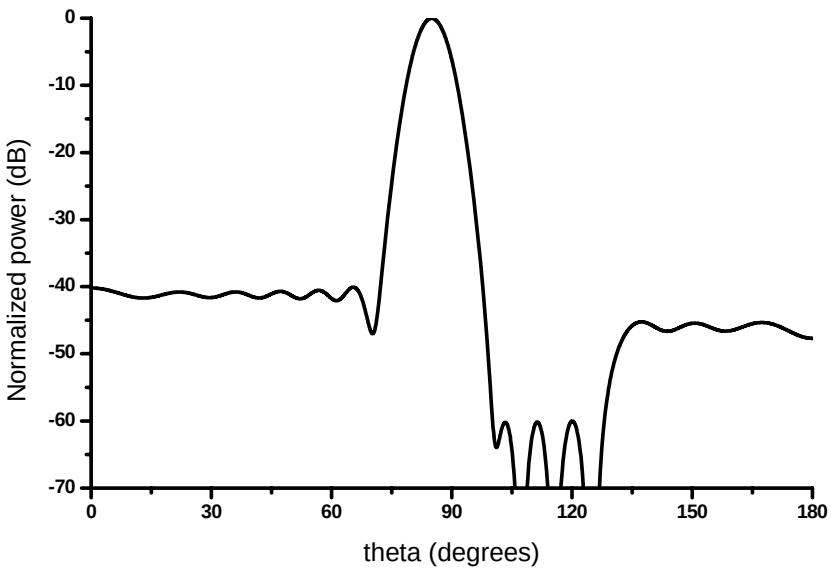
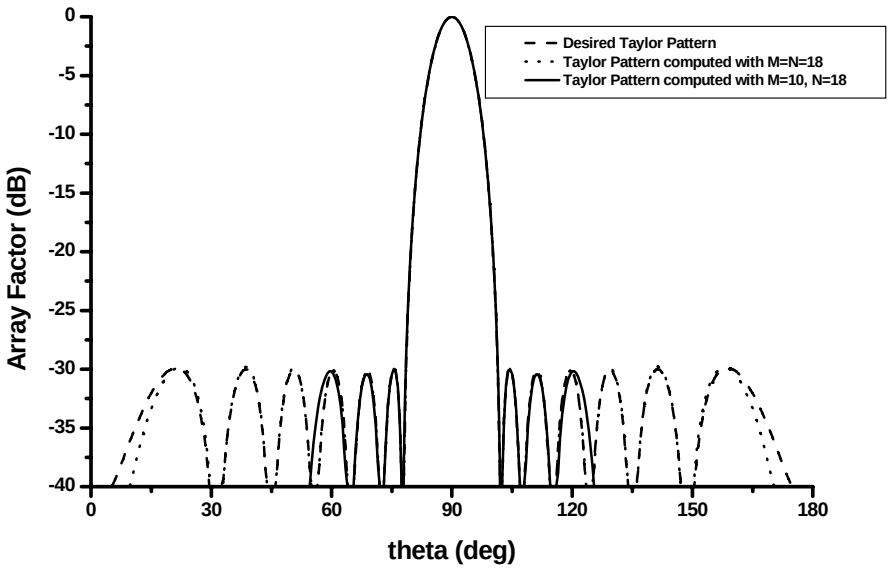


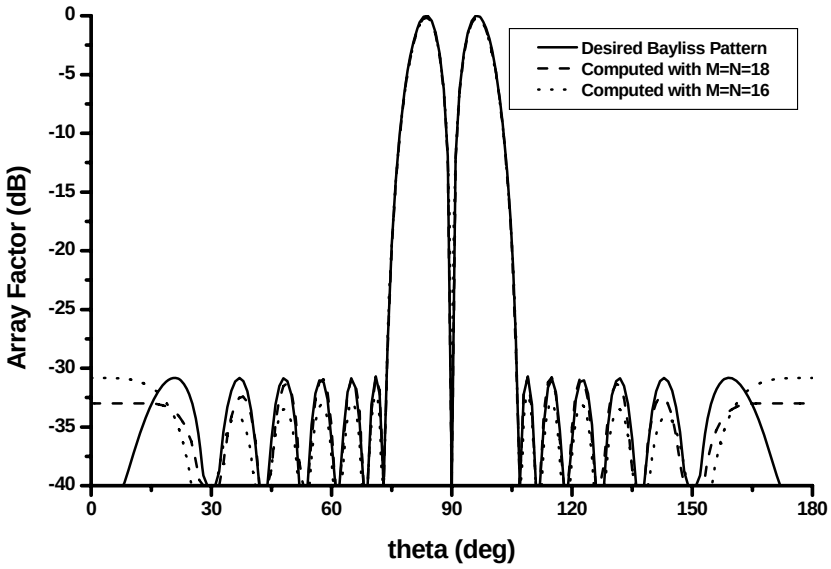

Figure 7. Desired and Computed Array Factor of an array with $d$=0.45$\lambda$, corresponding to a desired pattern of the form $\exp\left[k\left(\theta-\theta_o/2\right)^2\right]$ for the main beam.

A 32 element linear array with required maximum at $\theta_o = 85^o$ and different levels of the sidelobes in the left (≤-40dB) and right (≤-60dB and ≤-45dB) regions of the mainlobe has been designed. Figure 7 shows the pattern of the array with $d = 0.45\lambda$. It is noticed that this case can be compared to that given by Elliot et al [4], where the sidelobe level of the array is controlled.

Another useful example of a Taylor and a Bayliss pattern is given for $N$=18 elements, SLL=-40dB for the close and SLL=-30dB for the other sidelobes, and $d = 0.5\lambda$. Figure 8a shows the Taylor pattern for $M$=18 and $M$=10. It is obvious that for a lower number of $M$ the resulting pattern has a smaller number of sidelobes. Similar results are taken for the Bayliss pattern. Figure 8b shows the pattern for two different combinations of $M$ and $N$. One is with $M$=$N$=16 and the other is with $M$=$N$=18. For both cases the results are acceptable.




(a)




(b)

Figure 8. Computed Array Factor of an array with $d$=0.5$\lambda$, corresponding to (a) a Taylor desired pattern, and (b) a Bayliss desired pattern.

## 4. CONCLUSION

The method of orthosynthesis has been presented in this paper. It was supposed that the desired pattern comes from a set of $N$ element, uniformly illuminated arrays. The factor of each uniform array is the basis function of a non orthogonal M-dimensional vector space. The basis is orthonormalized by the Gram – Schmidt

procedure and the current of the array is found by the well-known orthogonal method. Several examples for different arrays and different patterns have shown the applicability of the method.

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