THE $L(c,n)$ NUMBERS AND THEIR APPLICATION TO THE ANALYSIS OF THE CIRCULAR FERRITE WAVEGUIDE PHASE SHIFTER

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Abstract

The finite positive real numbers $L(c,n)$ *(the common limits for* $k_- \to -\infty$ *of the infinite sequences of positive real numbers* $\{K_-(c,n,k_-)\}$ *and* $\{M_-(c,n,k_-)\}$ *where* $K_-(c,n,k_-) = |k_-|\zeta_{k_-,n}^{(c)}$, $M_-(c,n,k_-) = |a_-|\zeta_{k_-,n}^{(c)}$ *and* $\zeta_{k_-,n}^{(c)}$ *is the* n*th positive purely imaginary zero of the Kummer confluent hypergeometric function* $\Phi(a_-,c;x)$ *in* x *(*$n = 1,2,3...$*) with* $a_- = c/2 - jk_-$ *– complex,* c *– real,* $c \neq l$, $l = 0,-1,-2,-3,...$, $x = jz$ *– positive purely imaginary,* z *– real, positive,* k_- *– real, negative, or the numbers* $L(2-l,n)$ *in case* $c = l$*) are tabulated for positive and negative values of parameter* c *and for the first several natural numbers* n*. Based on the outcomes, some of their properties are studied. An approximate method is developed for computation of the differential phase shift, produced by the circular waveguide with azimuthally magnetized ferrite, propagating normal* TE_{01} *mode. The approach uses the elements* A, B *and* C *of the defined in a special way column matrix* $\boldsymbol{T}$*, the approximate values of which are determined by means of the* $L(c,n)$ *numbers. The error in the results obtained is less than one percent.*

1. INTRODUCTION

The L numbers sprang up into existence in the theory of latching phase shifters, built on circular ferrite and ferrite-dielectric waveguides with azimuthal magnetization and operating in the normal TE_{01} mode, worked out through the complex confluent

hypergeometric functions (CHFs) [1-9] or by means of them and besides utilizing the real cylindrical ones for the geometries of the second kind [1]. The same happened when these functions (all real) were harnessed to describe slow TE_{0n} modes in waveguides of the type mentioned [6,10,11]. The quantities considered have been defined as limits of sequences of real numbers whose terms involve the purely imaginary (real) zeros of specific complex (real) Kummer CHF [2,3,6,8] or the purely imaginary (real) roots of some transcendental equations, written by complex (real) CHFs [5,7,9,11] or except by them, addressing the real ordinary and modified derivative difference Bessel functions [1,10] of specially chosen parameters, as well [1-3,5-11]. The aforesaid functions are taken in the Kummer-Tricomi form [2,3,5-19], respectively in the Sovetov-Averbuch one [1,10,20]. The sequences tend to the limits in question when the imaginary part of complex first parameter (the real first parameter) of the complex (real) CHF(s) gets infinitely large negative [2,3,5-11] or both negative and positive [1]. In view of this, several categories of L numbers have been inaugurated [1-3,5-11]. As an eligible basis for their classification could serve the generating equations from which they stem from, enumerated above.

To get the differential phase shift, provided by the structures under study in case of normal propagation, is a serious task for whose solution various schemes have been proposed [1,3-8,9,21-34]. The ones, using the L numbers [3,4,8], avail of the traced by them peculiar envelope curves in the phase diagrams of the configurations at which the characteristics for negative (both negative and positive) magnetization terminate [1-5,6]. It seems that these new objects would find application also in plenty of other fields in which CHFs are employed [6]. Therefore, the knowledge of a few of their values only [1-3,5-9] obviously is not sufficient.

The discussion here is focussed on one representative of the class of numbers treated – the $L(c,n)$ ones, connected with the complex Kummer CHF $\Phi(a,c;x)$, provided $a = c/2 - jk$ is complex, c and k are real, $x = jz$ is positive purely imaginary and z is real, positive [2-4,8]. Tables of some of their values are compiled, depending on the second parameter c of $\Phi(a,c;x)$ and the number n of its positive purely imaginary zero in x. In addition, the paper reviews and extends the approximate method for calculation of the differential phase shift, due to the circular

waveguide, entirely filled with ferrite [3,4,8] which is an example of the putting into practice of these numbers.

2. $L(c,n)$ NUMBERS

Definition 1**:** *The common limits for* $k_- \to -\infty$ *of the infinite sequences of positive (negative) real numbers* $\{K_-(c,n,k_-)\}$ *and* $\{M_-(c,n,k_-)\}$ *where* $K_-(c,n,k_-) = |k_-|\zeta_{k_-,n}^{(c)}$, $M_-(c,n,k_-) = |a_-|\zeta_{k_-,n}^{(c)}$ *and* $\zeta_{k,n}^{(c)}$ *is the* n*th positive (negative) purely imaginary zero of the Kummer confluent hypergeometric function* $\Phi(a,c;x)$ *in* x *(*$n = 1,2,3...$*) with* $a = c/2 - jk$ *(*$a = c/2 + jk$*) – complex,* $c = 2\operatorname{Re} a$ *– a restricted positive or negative real number, different from zero or negative integer (*$c \neq l$*,* $l = 0,-1,-2,-3,...$*),* $x = jz$ *– positive (negative) purely imaginary,* z *– real, positive (negative) and* k *– real, (*c*,*n*– fixed), are finite positive (negative) real numbers, called* $L(c,n)$ *ones. It holds:*

$$\lim_{k_- \to -\infty} K_-(c,n,k_-) = \lim_{k_- \to -\infty} M_-(c,n,k_-) = L(c,n). \tag{1}$$

If $c = l$*, it is postulated:*

$$L(c,n) = L(2-l,n). \tag{2}$$

Moreover, it is valid:

$$\operatorname{sgn} L(c,n) = \operatorname{sgn} z. \tag{3}$$

(The subscript "–" indicates quantities, corresponding to negative k*.)*

Properties: *i) if* $c = l$ $(-\infty < l \le 0)$ *and* $n = 1$*, it holds* $L(l+0,1) = \lim_{c \to l+0} L(c,1) = 0$*; ii) in case* $c = l$ $(-\infty < l \le 0)$ *and* $n \ge 1$*, then* $L(l-0,n) < L(2-l,n) < L(l+0,n+1)$ *and* $L(l-0,n) = \lim_{c \to l-0} L(c,n) = \lim_{c \to l+0} L(c,n+1) = L(l+0,n+1) = L(2-l,n)$*; accordingly, it is accepted* $L(c,n) = L(2-l,n)$ *and hence* $L(c-0,n) < L(c,n) < L(c+0,n+1)$*; iii) assuming* $l < c_1 < c_2 < l+1 \le 0$ $(-\infty < l \le -1)$ *and* $n = 1$*, it is true*

$0 < L(c_1,1) < L(c_2,1) < L(1-l,1) < +\infty$; *iv) provided* $l < c_1 < c_2 < l+1 \le 0$ $(-\infty < l \le -1)$ *and* $n \ge 1$ $(n > 1)$, *it is valid* $0 < L(2-l,n) < L(c_1,n+1) < L(c_2,n+1) < L(1-l,n+1) < +\infty$ $(0 < L(2-l,n-1) < L(c_1,n) < L(c_2,n) < L(1-l,n) < +\infty)$;

Table 1. Values of the $L(c,n)$ numbers in case $n=1$ as a function of c for $c = -10$ (0.1) 0.

c	$L(c,n)$	c	$L(c,n)$	c	$L(c,n)$	c	$L(c,n)$	c	$L(c,n)$
-10.0		-8.0		-6.0		-4.0		-2.0	
-9.9	26.19089 68118	-7.9	17.14343 63819	-5.9	10.03047 09169	-3.9	4.83919 24332	-1.9	1.54771 56128
-9.8	30.68438 14359	-7.8	20.52534 23509	-5.8	12.39210 23191	-3.8	6.28953 77433	-1.8	2.22586 15078
-9.7	34.19201 67435	-7.7	23.19622 09888	-5.7	14.28929 19375	-3.7	7.48762 91585	-1.7	2.81862 52747
-9.6	37.25742 36635	-7.6	25.54918 81844	-5.6	15.98019 05547	-3.6	8.57559 78055	-1.6	3.37709 13769
-9.5	40.06788 59641	-7.5	27.72016 92830	-5.5	17.55453 33182	-3.5	9.60327 79174	-1.5	3.91946 53304
-9.4	42.71219 89839	-7.4	29.77369 14664	-5.4	19.05499 50654	-3.4	10.59442 84989	-1.4	4.45444 70199
-9.3	45.24001 43898	-7.3	31.74584 03450	-5.3	20.50544 09140	-3.3	11.56232 85784	-1.3	4.98687 39620
-9.2	47.68211 47368	-7.2	33.65897 42670	-5.2	21.92064 11208	-3.2	12.51517 54687	-1.2	5.51970 18426
-9.1	50.05899 40617	-7.1	35.52796 22185	-5.1	23.31039 98436	-3.1	13.45838 85151	-1.1	6.05485 68064
-9.0		-7.0		-5.0		-3.0		-1.0	
-8.9	21.42474 80709	-6.9	13.34580 99511	-4.9	7.19567 87737	-2.9	2.95799 04285	-0.9	0.60135 94011
-8.8	25.35186 94595	-6.8	16.20524 19397	-4.8	9.08658 52829	-2.8	4.00209 21312	-0.8	0.96338 27690
-8.7	28.43379 55770	-6.7	18.48078 63447	-4.7	10.62396 13301	-2.7	4.88406 83761	-0.7	1.29967 28148
-8.6	31.13711 06797	-6.6	20.49595 85652	-4.6	12.00529 35871	-2.6	5.69683 88860	-0.6	1.62888 38548
-8.5	33.62282 54808	-6.5	22.36288 50229	-4.5	13.29948 64798	-2.5	6.47320 30095	-0.5	1.95774 11153
-8.4	35.96732 35229	-6.4	24.13483 78916	-4.4	14.53934 61859	-2.4	7.22882 15474	-0.4	2.28943 38223
-8.3	38.21331 87133	-6.3	25.84160 94350	-4.3	15.74321 62589	-2.3	7.97243 67100	-0.3	2.62570 07727
-8.2	40.38728 53376	-6.2	27.50165 02011	-4.2	16.92243 85819	-2.2	8.70942 88338	-0.2	2.96757 32740
-8.1	42.50684 04662	-6.1	29.12722 05506	-4.1	18.08452 98622	-2.1	9.44333 92952	-0.1	3.31569 73770
-8.0		-6.0		-4.0		-2.0		0.0	

Table 2. Values of the $L(c,n)$ numbers in case $n=1$ as a function of c for $c = 0$ (0.1) 10.

c	$L(c,n)$	c	$L(c,n)$	c	$L(c,n)$	c	$L(c,n)$	c	$L(c,n)$
0.0		2.0	3.67049 26605	4.0	10.17661 64546	6.0	19.23473 20834	8.0	30.72690 00509
0.1	0.10492 12125	2.1	3.93219 59979	4.1	10.57021 34428	6.1	19.75223 46405	8.1	31.36404 87572
0.2	0.21940 16817	2.2	4.20080 97355	4.2	10.97014 77496	6.2	20.27580 31408	8.2	32.00709 95948
0.3	0.34307 65127	2.3	4.47628 94938	4.3	11.37640 14581	6.3	20.80542 75418	8.3	32.65604 60024
0.4	0.47564 39874	2.4	4.75859 34562	4.4	11.78895 72575	6.4	21.34109 80404	8.4	33.31088 15398
0.5	0.61685 02751	2.5	5.04768 21391	4.5	12.20779 84109	6.5	21.88280 50643	8.5	33.97159 98843
0.6	0.76647 86833	2.6	5.34351 81897	4.6	12.63290 87247	6.6	22.43053 92631	8.6	34.63819 48272
0.7	0.92434 18908	2.7	5.64606 62070	4.7	13.06427 25200	6.7	22.98429 15001	8.7	35.31066 02703
0.8	1.09027 62038	2.8	5.95529 25831	4.8	13.50187 46115	6.8	23.54405 28447	8.8	35.98899 02232
0.9	1.26413 72189	2.9	6.27116 53606	4.9	13.94570 02732	6.9	24.10981 45646	8.9	36.67317 87998
1.0	1.44579 64907	3.0	6.59365 41068	5.0	14.39573 52258	7.0	24.68156 81193	9.0	37.36322 02159
1.1	1.63513 89281	3.1	6.92272 97992	5.1	14.85196 56107	7.1	25.25930 51529	9.1	38.05910 87857
1.2	1.83206 07322	3.2	7.25836 47232	5.2	15.31437 79709	7.2	25.84301 74881	9.2	38.76083 89198
1.3	2.03646 77415	3.3	7.60053 23800	5.3	15.78295 92328	7.3	26.43269 71197	9.3	39.46840 51223
1.4	2.24827 40891	3.4	7.94920 74022	5.4	16.25769 66889	7.4	27.02833 62093	9.4	40.18180 19881
1.5	2.46740 11003	3.5	8.30436 54786	5.5	16.73857 79813	7.5	27.62992 70794	9.5	40.90102 42008
1.6	2.69377 63812	3.6	8.66598 32842	5.6	17.22559 10863	7.6	28.23746 22083	9.6	41.62606 65304
1.7	2.92733 30563	3.7	9.03403 84181	5.7	17.71872 43005	7.7	28.85093 42249	9.7	42.35692 38307
1.8	3.16800 91269	3.8	9.40850 93450	5.8	18.21796 62268	7.8	29.47033 59030	9.8	43.09359 10375
1.9	3.41574 69275	3.9	9.78937 53429	5.9	18.72330 57617	7.9	30.09566 01617	9.9	43.83606 31663
2.0	3.67049 26605	4.0	10.17661 64546	6.0	19.23473 20834	8.0	30.72690 00509	10.0	44.58433 53104

v) on condition that $0 < c_1 < c_2 < +\infty$ *and* $n = 1$, *it is fulfilled* $0 < L(c_1,1) < L(c_2,1) < +\infty$ *;*

Table 3. Values of the $L(c,n)$ numbers in case $n = 1$ as a function of c for c = 0 (0.5) 40.

c	$L(c,n)$	c	$L(c,n)$	c	$L(c,n)$	c	$L(c,n)$
0.0		10.0	44.58433 53104	20.0	148.08759 86828	30.0	306.90308 28311
0.5	0.61685 02751	10.5	48.41252 97318	20.5	154.72878 39190	30.5	316.27029 98591
1.0	1.44579 64907	11.0	52.38503 00316	21.0	161.50776 17906	31.0	325.77248 27607
1.5	2.46740 11003	11.5	56.50129 98266	21.5	168.42434 12537	31.5	335.40952 95278
2.0	3.67049 26605	12.0	60.76083 92651	22.0	175.47833 83511	32.0	345.18134 07560
2.5	5.04768 21391	12.5	65.16318 11682	22.5	182.66957 57974	32.5	355.08781 95392
3.0	6.59365 41068	13.0	69.70788 77133	23.0	189.99788 25979	33.0	365.12887 13704
3.5	8.30436 54786	13.5	74.39454 75648	23.5	197.46309 36963	33.5	375.30440 40471
4.0	10.17661 64546	14.0	79.22277 33791	24.0	205.06504 96491	34.0	385.61432 75825
4.5	12.20779 84109	14.5	84.19219 96232	24.5	212.80359 63245	34.5	396.05855 41200
5.0	14.39573 52258	15.0	89.30248 06575	25.0	220.67858 46229	35.0	406.63699 78578
5.5	16.73857 79813	15.5	94.55328 90451	25.5	228.68987 02173	35.5	417.34957 49633
6.0	19.23473 20834	16.0	99.94431 40526	26.0	236.83731 33121	36.0	428.19620 35109
6.5	21.88280 50643	16.5	105.47526 03165	26.5	245.12077 84178	36.5	439.17680 34083
7.0	24.68156 81193	17.0	111.14584 66523	27.0	253.54013 41417	37.0	450.29129 63322
7.5	27.62992 70794	17.5	116.95580 49883	27.5	262.09525 29919	37.5	461.53960 56665
8.0	30.72690 00509	18.0	122.90487 94055	28.0	270.78601 11938	38.0	472.92165 64433
8.5	33.97159 98843	18.5	128.99282 52735	28.5	279.61228 85190	38.5	484.43737 52862
9.0	37.36322 02159	19.0	135.21940 84682	29.0	288.57396 81242	39.0	496.08669 03560
9.5	40.90102 42008	19.5	141.58440 46628	29.5	297.67093 64003	39.5	507.86953 13040
10.0	44.58433 53104	20.0	148.08759 86828	30.0	306.90308 28311	40.0	519.78582 92130

Table 4. Values of the $L(c,n)$ numbers for c = 1,2,…,10 and n = 1,2,…,10.

n \ c	1	2	3	4	5
1	1.44579 64907	3.67049 26605	6.59365 41068	10.17661 64546	14.39573 52258
2	7.61781 55859	12.30461 40804	17.71249 97298	23.81939 31360	30.60694 90162
3	18.72175 16977	25.87486 34738	33.75517 72165	42.34886 24565	51.64245 25942
4	34.76007 11066	44.38019 17035	54.73004 72864	65.80021 35638	77.58056 49660
5	55.73307 59044	67.82041 35682	80.63877 90731	94.18134 98580	108.44028 40985
6	81.64083 82331	96.19547 62757	111.48189 11343	127.49491 89688	144.22832 47328
7	112.48338 21295	129.50536 02529	147.25958 79284	165.74202 20937	184.94759 38254
8	148.26071 74130	167.75005 69057	187.97196 34525	208.92318 62182	230.59952 66790
9	188.97284 86950	210.92956 19842	233.61906 58722	257.03868 99722	281.18490 14754
10	234.61977 83689	259.04387 31927	284.20092 19539	310.08869 21487	336.70417 09619
n \ c	6	7	8	9	10
1	19.23473 20834	24.68156 81193	30.72690 00509	37.36322 02159	44.58433 53104
2	38.06028 83854	46.16720 18348	54.91750 16696	64.30255 02495	74.31492 00694
3	61.62386 65333	72.28247 08239	83.60892 14637	95.59497 37578	108.23330 94262
4	90.06137 04799	103.23361 85656	117.08907 44284	131.62024 25129	146.82029 86715
5	123.40765 80609	139.07584 47506	155.43765 71909	172.48638 78485	190.21580 17214
6	161.67556 28050	179.83013 81033	198.68577 66895	218.23650 13486	238.47665 84946
7	204.87081 89278	225.50612 00984	246.84799 87124	268.89112 50223	291.63038 28414
8	252.99633 02328	276.10876 62028	299.93198 96958	324.46123 51825	349.69186 95753
9	306.05370 41196	331.64087 91788	357.94213 32949	384.95318 97156	412.66984 43237
10	364.04389 21667	392.10414 30617	420.88109 74325	450.37090 16909	480.56973 07465

vi) stipulating that $0 < c_1 < c_2 < +\infty$ *and* $n \geq 1$ $(n > 1)$, *results in* $0 < L(2,n) < L(c_1, n+1) < L(c_2, n+1) < +\infty$ $(0 < L(2, n-1) < L(c_1, n) < L(c_2, n) < +\infty)$*; vi) for any arbitrary finite real* c $(-\infty < c < +\infty)$, *it holds* $0 < L(c,n) < L(c, n+1) < +\infty$*;*

Table 5. Values of the $L(m,n)$ numbers for $m = -10$ (1) 10 and $n = 1,2,3,4$.

m	$L(m,n)$	$L(2-m,n)$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
-10	$L(-10,n)$	$L(12,n)$	60.76083 92651	96.19653 93944	135.44134 16716	179.20270 00315
-9	$L(-9,n)$	$L(11,n)$	52.38503 00316	84.94814 56903	121.51739 09178	162.68301 68941
-8	$L(-8,n)$	$L(10,n)$	44.58433 53104	74.31492 00694	108.23330 94262	146.82029 86715
-7	$L(-7,n)$	$L(9,n)$	37.36322 02159	64.30255 02495	95.59497 37578	131.62024 25129
-6	$L(-6,n)$	$L(8,n)$	30.72690 00509	54.91750 16696	83.60892 14637	117.08907 44284
-5	$L(-5,n)$	$L(7,n)$	24.68156 81193	46.16720 18348	72.28247 08239	103.23361 85656
-4	$L(-4,n)$	$L(6,n)$	19.23473 20834	38.06028 83854	61.62386 65333	90.06137 04799
-3	$L(-3,n)$	$L(5,n)$	14.39573 52258	30.60694 90162	51.64245 25942	77.58056 49669
-2	$L(-2,n)$	$L(4,n)$	10.17661 64546	23.81939 31360	42.34886 24565	65.80021 35638
-1	$L(-1,n)$	$L(3,n)$	6.59365 41068	17.71249 97298	33.75517 72165	54.73004 72864
0	$L(0,n)$	$L(2,n)$	3.67049 26605	12.30461 40804	25.87486 34738	44.38019 17035
1	$L(1,n)$	$L(1,n)$	1.44579 64907	7.61781 55859	18.72175 16977	34.76007 11066
2	$L(2,n)$	$L(0,n)$	3.67049 26605	12.30461 40804	25.87486 34738	44.38019 17035
3	$L(3,n)$	$L(-1,n)$	6.59365 41068	17.71249 97298	33.75517 72165	54.73004 72864
4	$L(4,n)$	$L(-2,n)$	10.17661 64546	23.81939 31360	42.34886 24565	65.80021 35638
5	$L(5,n)$	$L(-3,n)$	14.39573 52258	30.60694 90162	51.64245 25942	77.58056 49669
6	$L(6,n)$	$L(-4,n)$	19.23473 20834	38.06028 83854	61.62386 65333	90.06137 04799
7	$L(7,n)$	$L(-5,n)$	24.68156 81193	46.16720 18348	72.28247 08239	103.23361 85656
8	$L(8,n)$	$L(-6,n)$	30.72690 00509	54.91750 16696	83.60892 14637	117.08907 44284
9	$L(9,n)$	$L(-7,n)$	37.36322 02159	64.30255 02495	95.59497 37578	131.62024 25129
10	$L(10,n)$	$L(-8,n)$	44.58433 53104	74.31492 00694	108.23330 94262	146.82029 86715

vii) on the understanding that c_1 *and* c_2 *are arbitrary finite real numbers and* $-\infty < c_1 < c_2 < +\infty$, *it follows* $0 < L(c_1, n) < L(c_2, n+1) < +\infty$*; viii) provided* $c = m$, $m = 0, \pm 1, \pm 2, \ldots$, $(-\infty < m < +\infty)$, *it is true* $L(m,n) = L(2-m, n)$, *thus if* $m = l$, *then* $L(l,n) = L(2-l, n)$ *and in case* $m = -l$ $(m = p$, $p = 0,1,2,\ldots,)$, *it is valid* $L(-l,n) = L(2+l,n)$ $[L(p,n) = L(2-p,n)]$*; ix) in particular when* $m = 1$, *obviously* $L(1,n) \equiv L(1,n)$*; x) for any* L *there exist infinite different real numbers* c_l *(each of which satisfies the condition* $l < c_l < l+1$, $(-\infty < l \leq -1)$, *or* $l < c_l < +\infty$, $(l = 0)$*) and* n *(part or all of which are identical), for that the values of* $L(c_l, n)$ *are the same; xi) if* $L_1 < L_2$, *for the corresponding numbers* c_l *it is fulfilled* $c_{l_1} < c_{l_2}$*; xii) provided* $L(c_l, n) = L$ *and* $0 < L \leq L(2,1)$, *then* $n = 1$.

Properties of related quantities: *I) For any real* c*, including the case* $c=l$ *which is equivalent to the one* $c=2-l$*, the following is valid: i) when* $k=0$*, then* $\zeta_{0,n}^{(c)}=2\nu_{(c-1)/2,n}$*,* $K(c,n,0)=0$ *and* $M(c,n,0)=c\zeta_{0,n}^{(c)}/2$ *[*$M(c,n,0)=c\nu_{(c-1)/2,n}$*],* *[*$\nu_{(c-1)/2,n}$ *– nth zero of Bessel function* $J_{(c-1)/2}(z/2)$*] ; ii) assuming* $k_-\to-\infty$*, it holds* $\lim_{k_-\to-\infty}\zeta_{k_-,n}^{(c)}=0$*; iii) if* k_{1-} *and* k_{2-} *are any two large negative real numbers,* $|a_{1-}|$ *and* $|a_{2-}|$ *are the moduli of complex first parameter of* $\Phi(a,c;x)$*, relevant to them, and* $\zeta_{k_{1-},n}^{(c)}$ *and* $\zeta_{k_{2-},n}^{(c)}$ *are the corresponding purely imaginary zeros of the function for the same* n*, it is true: 1)* $\zeta_{k_{1-},n}^{(c)}\approx 10^{-h}\zeta_{k_{2-},n}^{(c)}$ *(if* $|k_{1-}|=10^h|k_{2-}|$ *and* h *is a positive or negative integer or zero); 2)* $k_{1-}/k_{2-}\approx\zeta_{k_{2-},n}^{(c)}/\zeta_{k_{1-},n}^{(c)}$*; 3)* $\zeta_{k_{1-},n}^{(c)}/k_{2-}\approx\zeta_{k_{2-},n}^{(c)}/k_{1-}$*; 4)* $|k_{1-}|\zeta_{k_{1-},n}^{(c)}\approx|k_{2-}|\zeta_{k_{2-},n}^{(c)}\approx L(c,n)$*; 5)* $|a_{1-}|/|a_{2-}|\approx\zeta_{k_{2-},n}^{(c)}/\zeta_{k_{1-},n}^{(c)}$*; 6)* $\zeta_{k_{1-},n}^{(c)}/|a_{2-}|\approx\zeta_{k_{2-},n}^{(c)}/|a_{1-}|$*; 7)* $|a_{1-}|\zeta_{k_{1-},n}^{(c)}\approx|a_{2-}|\zeta_{k_{2-},n}^{(c)}\approx L(c,n)$*; iv) in case* k_{1-} *and* k_{2-} *are any two large negative real numbers, then* $K_-(c,n,k_{1-})\approx K_-(c,n,k_{2-})\approx M_-(c,n,k_{1-})\approx M_-(c,n,k_{2-})\approx L(c,n)$*; v) for any large negative* k_- *it holds* $K_-(c,n,k_-)<L(c,n)<M_-(c,n,k_-)$*; vi) provided* $|k_-|$ *is large, it is fulfilled: 1)* $\zeta_{k_-,n}^{(c)}=L(c,n)/|k_-|$*; 2)* $\zeta_{k_-,n}^{(c)}/|k_-|\approx L(c,n)/|k_-|^2$*; 3)* $\zeta_{k,n}^{(c)}\approx L(c,n)/|a|$*; 4)* $\zeta_{k,n}^{(c)}/|a|\approx L(c,n)/|a|^2$*; II) If* $c>0$*, in case* $k_+\to+\infty$*, it is valid* $\lim_{k_+\to+\infty}\zeta_{k_+,n}^{(c)}=\lim_{k_+\to+\infty}K_+(c,n,k_+)=\lim_{k_+\to+\infty}M_+(c,n,k_+)=+\infty$ *where* $K_+(c,n,k_+)=|k_+|\zeta_{k_+,n}^{(c)}$*,* $M_+(c,n,k_+)=|a_+|\zeta_{k_+,n}^{(c)}$ *(The subscript "+" indicates quantities, corresponding to positive* k*.)*

General characteristics: i) When c rises *from* l to $l+1$, $(-\infty<l\le-1)$, the $L(c,n)$ numbers strictly monotonously increase from 0 to $L(1-l,1)$, if $n=1$ and from $L(2-l,n-1)$ to $L(1-l,n)$, on the understanding that $n>1$. Provided c grows from 0 to $+\infty$ (remaining finite), they mount in the same interval in case $n=1$ and in the one ($L(2,n-1),+\infty$) for $n>1$ (remaining finite). At the points $c=l$ jumps from $L(2-l,1)$ to 0, assuming $n=1$ and from $L(2-l,n)$ to $L(2-l,n-1)$,

stipulating that $n>1$ are observed. ii) The sequence of real numbers 0, $L(c,1)$, $L(2-(l+1),1)$, $L(c,2)$, $L(2-(l+2),2)$, $L(c,3)$, $L(2-(l+3),3)$, …, $L(2-(l+(-l-1)),-l-1)$, $L(c,-l)$, $L(2,-l)$, $L(c,-l+1)$ is strictly monotonously increasing, on condition that c enlarges from l ($-\infty<l$) to $+\infty$ (remaining finite). iii) At fixed n the numbers $L(m,n)$ are mirror images of the ones $L(2-m,n)$ in the straight line $m=1$. $L(1,n)$ coincides with its image. iv) The infinite sequence of real numbers $\{l-c_l\}$ is convergent for $l\to-\infty$ and its limit is zero.

Comments: The $L(c,n)$ numbers have been introduced for the first time in Ref. [2] and the $K_{\pm}(c,n,k_{\pm})$ and $M_{\pm}(c,n,k_{\pm})$ ones – in Ref. [8]. In the general case (c – arbitrary real) Definition 1 is substantiated by Lemma 1 in Refs. [3,8], whereas in the special one (c – a positive integer) this is done through Lemma 1 in Ref. [2]. Tables 1, 2 and 3 list $L(c,n)$ for c varying from –10 to 0 and from 0 to 10 with a step 0.1, and from 0 to 40 with a step 0.5, resp., when $n=1$. Table 4 gives them for values of the natural numbers c = 1 to 10 and n = 1 to 10. The computations are performed by means of the representation of $\Phi(a,c;x)$ around the point 0 in terms of an infinite power series [2,3,6,8,11-19] with parameters – specified in Definition 1, accepting k = -1.10^{10}. The approach does not allow to obtain $L(c,n)$ for $c=l$ where $\Phi(a,c;x)$ does not exist (cf. the empty places in Tables 1-3). This is effectuated with the help of eqn. (2) whose proof is given in Refs. [3,8]. The more general case $c=m$ is illustrated in Table 5. It is assumed here that $\operatorname{sgn} L(c,n)=+1$ ($\operatorname{sgn} z=+1$). All complementary results, given above, are due to the numerical analysis of numbers advanced and of the related with them quantities. Part of the same are based on the study of the values, presented in Tables 1-5. Some of the properties of the zeros $\zeta_{k,n}^{(c)}$ have been established in Ref. [2] and of the $K_{\pm}(c,n,k_{\pm})$ and $M_{\pm}(c,n,k_{\pm})$ numbers – in Ref. [8].

3. AZIMUTHALLY MAGNETIZED CIRCULAR FERRITE WAVEGUIDE PHASE SHIFTER

Definition 2: *The set of points* $R = R(c, n, \bar{\beta}, \bar{r}_0)$ *constitutes a* ***space of phase characteristics*** $\mathcal{R}$ *of the ferrite-loaded circular waveguide with azimuthal magnetization, propagating normal* TE_{0n} *mode, if their co-ordinates* $\bar{\beta}$ *and* $\bar{r}_0$ *are connected through the equations:*

$$\Phi(a, c; x_0) = 0 \tag{4}$$

in which $a = c/2 - jk$, $c = 3$, $x_0 = jz_0$, $k = \alpha\bar{\beta}/(2\bar{\beta}_2)$, $\bar{\beta}_2 = (1 - \alpha^2 - \bar{\beta}^2)^{1/2}$, $z_0 = 2\bar{\beta}_2\bar{r}_0$, ($k$, z_0 – *real*, $-\infty < k < +\infty$, $z_0 > 0$), $\bar{\beta} = \beta/(\beta_0\sqrt{\varepsilon_r})$, $\bar{\beta}_2 = \beta_2/(\beta_0\sqrt{\varepsilon_r})$, $\bar{r}_0 = \beta_0 r_0\sqrt{\varepsilon_r}$, $\beta_0 = \omega\sqrt{\varepsilon_0\mu_0}$, $\alpha = \gamma M_r/\omega$ *is the off-diagonal element of Polder permeability tensor of the lossless ferrite,* $-1 < \alpha < 1$, (γ – *gyromagnetic ratio,* M_r – *remanent magnetization,* ω – *angular frequency of the wave),* β *is the phase constant,* β_2 *is the radial wavenumber,* ε_r *is the scalar ferrite relative permittivity,* r_0 *is the waveguide radius, and*

$$\bar{\beta}_2 = \zeta_{k,n}^{(c)}/(2\bar{r}_0) \tag{5}$$

in that $\zeta_{k,n}^{(c)}$ *are the* n*th roots of eqn. (4) in* x_0 *(in* z_0*),* $n = 1,2,3,\ldots$, (k *is a varying and* α – *a discrete parameter).*

Definition 3: The co-*ordinates* $\bar{\beta}$ and $\bar{r}_0$ of the points of **space of phase characteristics** $\mathcal{R}$ for the normal TE_{0n} mode suffice also the equations:

$$\bar{r}_{0\pm}(c, n, \alpha_\pm, k_\pm) = [K_\pm(c, n, k_\pm)/|\alpha_\pm|]\left\{\left[1 + (\alpha_\pm/(2k_\pm))^2\right]/(1 - \alpha_\pm^2)\right\}^{1/2}, \tag{6}$$

$$\bar{\beta}_\pm(\alpha_\pm, k_\pm) = \left\{(1 - \alpha_\pm^2)/\left[1 + (\alpha_\pm/(2k_\pm))^2\right]\right\}^{1/2} \tag{7}$$

in which α is the off-diagonal element of ferrite permeability tensor, $\bar{r}_0$ is the normalized guide radius and $K_{\pm}(c,n,k_{\pm})$ are certain positive real numbers, with $c=3$, $n=1,2,3,...$, (k is a varying and α – a discrete parameter). (The subscripts "+", "–" indicate quantities, related to positive ($\alpha>0$) or negative ($\alpha<0$) magnetization, resp.) Thus, it could be written $R=R(c,n,\alpha_{\pm},k_{\pm})$.

Definition 4: *The ensemble of points $P=P(c,n,|\alpha|,\bar{r}_0)$ forms a* ***<u>space of phase shifter operation</u>*** *$\mathcal{P}$ of the azimuthally magnetized circular ferrite waveguide, working in the normal TE_{0n} mode, if their co-ordinates $|\alpha|$ and $\bar{r}_0$ satisfy the criterion:*

$$\nu_{(c-1)/2,n} < \bar{r}_0\sqrt{1-\alpha^2} < L(c,n)/|\alpha|, \tag{8}$$

or

$$M(c,n,0)/c < \bar{r}_0\sqrt{1-\alpha^2} < L(c,n)/|\alpha| \tag{9}$$

in which α is the off-diagonal element of ferrite permeability tensor, $\bar{r}_0$ is the normalized guide radius, $\nu_{(c-1)/2,n}$ is the nth root of Bessel function $J_{(c-1)/2}(z/2)$, $L(c,n)$ and $M(c,n,0)$ are certain positive real numbers in that $c=3$ and $n=1,2,3,...$. At each point $P\in\mathcal{P}$ a differential phase shift $\Delta\bar{\beta}=\Delta\bar{\beta}(c,n,|\alpha|,\bar{r}_0)$ is produced, symbolized in normalized notations as $\Delta\bar{\beta}=\bar{\beta}_-(c,n,|\alpha|,\bar{r}_0)-\bar{\beta}_+(c,n,|\alpha|,\bar{r}_0)$ where $\bar{\beta}_+$ and $\bar{\beta}_-$ are the normalized phase constants for positive (counterclockwise) and negative (clockwise) magnetization of the ferrite.

Comments: The problem for normal TE_{0n} modes propagation in the circular waveguide, uniformly filled with azimuthally magnetized ferrite, leading to eqns. (4), (5), is considered in Refs. [2-4,8,34]. Equations (6),(7) [2,8] allow to figure the phase characteristics $\bar{\beta}(\bar{r}_0)$ of the configuration *examined*. The phase shifter operation of the structure under investigation is discussed in Refs. [2-4,6,8], the notion ***<u>space of phase shifter (phaser) operation</u>*** $\mathcal{P}$ is introduced in Ref. [4], the one ***<u>space of phase characteristics</u>*** $\mathcal{R}$ is original. The criterion for phase shifter (phaser) operation for

TE_{01} is derived in Ref. [2] and substantiated in Ref. [4]. Its new form (9) in terms of $M(c,n,0)$ and $L(c,n)$ numbers is suggested recently in Ref. [8]. If $c=3$ and $n=1$, it holds: $\nu_{(c-1)/2,n}$ = 3.83170 59702, $M(c,n,0)$ = 11.49511 79106 and $L(c,n)$ = 6.59365 41068. In this case the parameters c and n in the notations for R, P, $\bar{r}_0$ and $\bar{\beta}$ could be dropped.

4. *A*, *B*, *C* NUMBERS. *T* MATRIX

Definition 5: *To each point* $P \in \mathcal{P}$ *corresponds a* ***<u>column matrix</u>***

$$\boldsymbol{T}\ (c,n,|\alpha|,\bar{r}_0)=\begin{pmatrix} A(c,n,|\alpha|,\bar{r}_0) \\ B(c,n,|\alpha|,\bar{r}_0) \\ C(c,n,|\alpha|,\bar{r}_0) \end{pmatrix}, \tag{10}$$

of the type 3×1 *with real elements, determined by the equation*

$$\boldsymbol{T}\ (c,n,|\alpha|,\bar{r}_0)=\boldsymbol{\Lambda}\ (c,n,|\alpha|,\bar{r}_0)\ \boldsymbol{U}\ (|\alpha|,\bar{r}_0) \tag{11}$$

in which $\boldsymbol{\Lambda}\ (c,n,|\alpha|,\bar{r}_0)=\left(\lambda_{ij}(c,n,|\alpha|,\bar{r}_0)\right)$ *is a scalar matrix of the type* 3×3 *with elements*

$$\lambda_{ij}(c,n,|\alpha|,\bar{r}_0)=\begin{cases} 0, & i\neq j \\ \left[\bar{\beta}_-(c,n,|\alpha|,\bar{r}_0)-\bar{\beta}_+(c,n,|\alpha|,\bar{r}_0)\right]\bar{r}_0/|\alpha|, & i=j \end{cases}, \tag{12}$$

$i,j=1,2,3$ *and* $\boldsymbol{U}\ (|\alpha|,\bar{r}_0)$ *is a column matrix of the type* 3×1, *composed in the following way:*

$$\boldsymbol{U}\ (|\alpha|,\bar{r}_0)=\begin{pmatrix} 1/\bar{r}_0 \\ |\alpha| \\ 1 \end{pmatrix}. \tag{13}$$

Here $\Delta\bar{\beta}(c,n,|\alpha|,\bar{r}_0)=\bar{\beta}_-(c,n,|\alpha|,\bar{r}_0)-\bar{\beta}_+(c,n,|\alpha|,\bar{r}_0)$ *is the normalized differential phase shift, provided by the azimuthally magnetized circular ferrite*

waveguide, operating in the normal TE_{0n} *mode,* α *is the off-diagonal element of ferrite permeability tensor,* $\bar{r}_0$ *is the normalized guide radius,* $\bar{\beta}_+$ *and* $\bar{\beta}_-$ *are the normalized phase constants for positive and negative magnetization of the ferrite,* $c = 3$ *and* $n = 1,2,3,\ldots$.

Definition 6: *The diagonal elements* $\lambda_{ii}(c,n,|\alpha|,\bar{r}_0)$ *of the scalar matrix* $\mathbf{\Lambda}\,(c,n,|\alpha|,\bar{r}_0)$ *could be presented also in the form*

$$\lambda_{ii}\left(c,n,|\alpha|,\begin{Bmatrix}k_+\\k_-\end{Bmatrix}\right) = [K_-(c,n,k_-) - K_+(c,n,k_+)]/|\alpha|^2, \tag{14}$$

(the matrix could be written in the shape $\mathbf{\Lambda}\left(c,n,|\alpha|,\begin{Bmatrix}k_+\\k_-\end{Bmatrix}\right)$*) where* $K_\pm(c,n,k_\pm)$ *are certain positive real numbers which meet the requirement*

$$K_-(c,n,k_-)/K_+(c,n,k_+) = \left\{\left[1+(|\alpha|/(2|k_-|))^2\right]/\left[1+(|\alpha|/(2|k_+|))^2\right]\right\}^{1/2}, \tag{15}$$

corresponding to the conditions $\bar{r}_{0+} = \bar{r}_{0-} = \bar{r}_0$ *and* $|\alpha_+| = |\alpha_-| = |\alpha|$, α *is the off-diagonal element of ferrite permeability tensor,* $\bar{r}_0$ *is the normalized guide radius of the azimuthally magnetized circular ferrite waveguide that sustains normal* TE_{0n} *modes,* c *and* k *are the second parameter and the imaginary part of the first one* a *(*$a = c/2 - jk$*) of wave function for propagation – the complex Kummer function* $\Phi(a,c;x)$ *and* n *is the number of its purely imaginary zero,* $i = 1,2,3$, $c = 3$ *and* $n = 1,2,3,\ldots$. *(The subscripts "+","–" correspond to positive, negative ferrite magnetization.)*

Definition 7: *The ensemble of column matrices* $\boldsymbol{T}$*, constructed according Definition 5 makes up a* ***space of matrices*** $\mathcal{T}$*. The matrices themselves can be considered as points in* $\mathcal{T}$ *and their elements (components) – the real numbers* A, B *and* C *as co-ordinates of these points.*

Comments: The real numbers A, B and C have been introduced as a means for computation of the differential *phase* shift, produced by the waveguide explored [34], serving as factors of proportionality in simple formulae, suggested for the purpose. Expressing them through the $K_{\pm}(c, n, k_{\pm})$ numbers is a quite recent idea [8]. It is believed that the notions ***matrix of numbers*** and ***space of matrices***, replacing the ones ***triad*** and ***space of triads*** [4] are more successful. To simplify the notations, relevant to the TE_{01} mode, the parameters c and n in all quantities save for $K_{\pm}(c, n, k_{\pm})$ could be omitted, as done earlier [4,8,34]. Eqns. (10), (11), (13) could be written in terms of $\bar{r}_0$, k_+ or k_- which are interchangeable in view of the relations (6), (15) between them.

5. $\boldsymbol{T}_{en-}$ AND $\boldsymbol{T}_{en-,\min}$ MATRICES

Let $k_- = k_{en-} \to -\infty$ and $\alpha = \alpha_{en-} < 0$. Then, eqns. (6), (7) *become*:

$$\bar{r}_{0en-}(c, n, |\alpha_{en-}|, k_{en-}) = L(c, n) / \left[|\alpha_{en-}| \left(1 - \alpha_{en-}^2\right)^{1/2} \right], \tag{16}$$

$$\bar{\beta}_{en-}(\alpha_{en-}, k_{en-}) = \left(1 - \alpha_{en-}^2\right)^{1/2} \tag{17}$$

which is the equation of the envelope curve of phase characteristics $\bar{\beta}_{en-}$ = $\bar{\beta}_{en-}(\bar{r}_{0en-})$ (the one, restricting the characteristics for negative magnetization from the side of higher frequencies) in parametric form with α_{en-} as parameter ($c = 3$ and $n = 1$ for normal TE_{01} mode) [2]. There is k_+ for which if $\alpha = \alpha_+ = |\alpha_{en-}| > 0$, it is true $\bar{r}_{0+} = \bar{r}_{0en-}$. The corresponding value of the phase constant for positive magnetization is $\bar{\beta}_+ = \bar{\beta}_+(c, n, |\alpha_{en-}|, k_+)$. The numbers k_+ and $\bar{\beta}_+$ searched are specified, harnessing an iterative procedure, described lately [4]. (All quantities, related to the envelope are denoted by the subscript "$en-$".)

Accordingly, for the matrices, connected with the envelope, it could be written:

$$\mathbf{\Lambda}_{en-}(c,n,|\alpha_{en-}|,k_+)=\left(\lambda_{en-,ij}(c,n,|\alpha_{en-}|,k_+)\right), \tag{18}$$

$$\lambda_{en-,ii}(c,n,|\alpha_{en-}|,k_+)=\begin{cases} L(c,n)\left\{\left[1-\bar{\beta}_+(c,n,|\alpha_{en-}|,k_+)/\left(1-|\alpha_{en-}|^2\right)^{1/2}\right]/|\alpha_{en-}|^2\right\} \\ \left[L(c,n)-K_+(c,n,k_+)\right]/|\alpha_{en-}|^2 \end{cases}, \tag{19}$$

$$L(c,n)/K_+(c,n,k_+)=\left\{1/\left[1+(|\alpha_{en-}|/(2|k_+|))^2\right]\right\}^{1/2}, \tag{20}$$

$$\boldsymbol{T}_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-})=\mathbf{\Lambda}_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-})\,\boldsymbol{U}_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-}), \tag{21}$$

$$\boldsymbol{T}_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-})=\begin{pmatrix} A_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-}) \\ B_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-}) \\ C_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-}) \end{pmatrix}, \tag{22}$$

$$\boldsymbol{U}_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-})=\begin{pmatrix} 1/\bar{r}_{0en-} \\ |\alpha_{en-}| \\ 1 \end{pmatrix} \tag{23}$$

where κ_{en-} stands for any of the interchangeable symbols $\bar{r}_{0en-}$, k_{en-} and the relevant k_+. If $|\alpha_{en-}|$ is accepted as a basic parameter, $\bar{r}_{0en-}$ is determined from eqn. (16) and κ_{en-} = k_{en-}. In case $\bar{r}_{0en-}=\bar{r}_{0+}$ is figured from eqn. (6) with $\alpha_+=|\alpha_{en-}|$, then κ_{en-} = k_+. The numerical values of $\bar{r}_{0en-}$, A_{en-}, B_{en-}, C_{en-} have been computed earlier as a function of $|\alpha_{en-}|$ = 0.1 (0.1) 0.9 and listed in Table 4 [4] and in Table 4 [8].

At the minimum of envelope with respect to $\bar{r}_{0en-}-\min\bar{r}_{0en-}=2L(c,n)$, attained at $\alpha_{en-,\min}=-1/\sqrt{2}$ and $\bar{\beta}_{en-,\min}=1/\sqrt{2}$ [4], eqns. (18) – (23) are recorded as:

$$\mathbf{\Lambda}_{en-,\min}\left(c,n,\left|\alpha_{en-,\min}\right|,k_{+,\min}\right)=\left(\lambda_{en-,\min,ij}\left(c,n,\left|\alpha_{en-,\min}\right|,k_{+,\min}\right)\right), \tag{24}$$

$$\lambda_{en-,\min,ii}\left(c,n,\left|\alpha_{en-,\min}\right|,k_{+,\min}\right)=$$

$$=\begin{cases} L(c,n)\left\{\left[1-\bar{\beta}_{+,\min}\left(c,n,\left|\alpha_{en-,\min}\right|,k_{+,\min}\right)/\left(1-\left|\alpha_{en-,\min}\right|^2\right)^{1/2}\right]/\left|\alpha_{en-,\min}\right|^2\right\} \\ \left[L(c,n)-K_{+,\min}\left(c,n,k_{+,\min}\right)\right]/\left|\alpha_{en-,\min}\right|^2 \end{cases}, \tag{25}$$

$$L(c,n)/K_{+,\min}(c,n,k_{+,\min}) = \left\{1/\left[1+\left(\left|\alpha_{en-,\min}\right|/\left(2\left|k_{+,\min}\right|\right)\right)^2\right]\right\}^{1/2}, \tag{26}$$

$$\boldsymbol{T}_{en-,\min}\left(c,n,\left|\alpha_{en-,\min}\right|,\kappa_{en-,\min}\right) =$$

$$\boldsymbol{\Lambda}_{en-,\min}\left(c,n,\left|\alpha_{en-,\min}\right|,\kappa_{en-,\min}\right)\boldsymbol{U}_{en-\min}\left(c,n,\left|\alpha_{en-,\min}\right|,\kappa_{en-,\min}\right), \tag{27}$$

$$\boldsymbol{T}_{en-,\min}\left(c,n,\left|\alpha_{en-,\min}\right|,\kappa_{en-,\min}\right) = \begin{pmatrix} A_{en-,\min}\left(c,n,\left|\alpha_{en-,\min}\right|,\kappa_{en-,\min}\right) \\ B_{en-,\min}\left(c,n,\left|\alpha_{en-,\min}\right|,\kappa_{en-,\min}\right) \\ C_{en-,\min}\left(c,n,\left|\alpha_{en-,\min}\right|,\kappa_{en-,\min}\right) \end{pmatrix}, \tag{28}$$

$$\boldsymbol{U}_{en-,\min}\left(c,n,\left|\alpha_{en-,\min}\right|,\kappa_{en-,\min}\right) = \begin{pmatrix} 1/\min\bar{r}_{0en-} \\ \left|\alpha_{en-,\min}\right| \\ 1 \end{pmatrix}. \tag{29}$$

Eqns. (25) and (29) could be simplified like:

$$\lambda_{en-,\min,ii}\left(c,n,\left|\alpha_{en-,\min}\right|,k_{+,\min}\right) =$$

$$= \begin{cases} 2L(c,n)\left\{\left[1-\sqrt{2}\bar{\beta}_{+,\min}\left(c,n,\left|\alpha_{en-,\min}\right|,k_{+,\min}\right)\right]\right\} \\ 2\left[L(c,n)-K_{+,\min}\left(c,n,k_{+,\min}\right)\right] \end{cases}, \tag{30}$$

$$\boldsymbol{U}_{en-,\min}\left(c,n,\left|\alpha_{en-,\min}\right|,\kappa_{en-,\min}\right) = \begin{pmatrix} 1/[2L(c,n)] \\ 1/\sqrt{2} \\ 1 \end{pmatrix}, \tag{31}$$

($\kappa_{en-,\min}$ is a general notation of the parameters $\min\bar{r}_{0en-}$, $k_{en-,\min} = -\infty$ and $k_{+,\min}$.) The numerical investigation in case of normal TE_{01} mode yields: $c = 3$, $n = 1$, $L(c,n)$ = 6.59365 41068, $\min\bar{r}_{0en-}$ = 13.18730 82136, $k_{+,\min}$ = 0.54841 53081, $\zeta^{(3)}_{k_{+,\min},1}$ = 10.10518 02318, $\bar{\beta}_{+,\min}$ = 0.59430 92282, $K_{+,\min}\left(c,n,k_{+,\min}\right)$ = 5.54183 55298. Correspondingly, for the matrix $\boldsymbol{T}_{en-,\min}$ it is calculated:

$$\boldsymbol{T}_{en-,\min}(3,1,0.70710\,67812,\ 13.18730\,82136) = \begin{pmatrix} 0.1595198292 \\ 1.4874960967 \\ 2.1036371539 \end{pmatrix}. \tag{32}$$

6. $\boldsymbol{T}_{approx}$, $\boldsymbol{T}_{approx,en-}$ AND $\boldsymbol{T}_{approx,en-,\min}$ MATRICES

The following contention and subsequent considerations are based on and constitute a generalization of the numerical and analytical study done in Ref. [4] and extended recently in Ref. [8].

Assumption for independence of A, B, C numbers of certain parameters: *It could be accepted that A and B are constants with regard to $|\alpha|$ and $\bar{r}_0$, resp., whereas C is constant in relation to both $|\alpha|$ and $\bar{r}_0$. Thus, it is assumed that:*

$$\boldsymbol{T}\,(c,n,|\alpha|,\bar{r}_0) \approx \boldsymbol{T}_{approx}(c,n,|\alpha|,\bar{r}_0) \tag{33}$$

where

$$\boldsymbol{T}_{approx}(c,n,|\alpha|,\bar{r}_0) = \begin{pmatrix} A_{approx}(c,n,\bar{r}_0) \\ B_{approx}(c,n,|\alpha|) \\ C_{approx}(c,n) \end{pmatrix}. \tag{34}$$

Accordingly, it has been shown that knowing the $\boldsymbol{T}$ matrix (anyone of its elements) at an arbitrary point $P(c,n,|\alpha|,\bar{r}_0)$, whose co-ordinates satisfy condition (8) or (9), it is possible to obtain its approximation $\boldsymbol{T}_{approx}$ at any other point of co-ordinates, subject to the same criterion with a fair degree of accuracy [4,8]. Several simple methods have been elaborated for this purpose [4,8]. In particular, if the matrix $\boldsymbol{T}_{en-}$ for arbitrary $|\alpha_{en-}|$ is known, the corresponding approximate one $\boldsymbol{T}_{approx,en-}$, relevant to any point, belonging to the space of phase shifter operation, is easily calculated from the equation:

$$\boldsymbol{T}_{approx,en-}(c,n,|\alpha|,\bar{r}_0) = \mu_{en-,i}(c,n,|\alpha_{en-}|,\kappa_{en-})\,\boldsymbol{U}\,(|\alpha|,\bar{r}_0) \tag{35}$$

in which $\mu_{en-,i}(c,n,|\alpha_{en-}|,\kappa_{en-})$ is any of the elements of the matrix:

$$\mathbf{M}_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-}) = \left(\mu_{en-,i}(c,n,|\alpha_{en-}|,\kappa_{en-})\right), \tag{36}$$

defined by the relation:

$$\mathbf{M}_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-})=[\boldsymbol{E}\ \boldsymbol{U}_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-})]^{-1}\boldsymbol{T}_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-}) \qquad (37)$$

in that $\boldsymbol{E}$ is a unit matrix of type 3×3 and $(\boldsymbol{E}\ \boldsymbol{U}_{en-})^{-1}$ is the reverse of the matrix $\boldsymbol{E}\ \boldsymbol{U}_{en-}$. In explicit form:

$$\mathbf{M}_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-})=\begin{pmatrix} A_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-})\bar{r}_{0en-}(c,n,|\alpha_{en-}|,\kappa_{en-}) \\ B_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-})/|\alpha_{en-}| \\ C_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-}) \end{pmatrix}. \qquad (38)$$

It could be proved that for the same set of parameters $(c,n,|\alpha_{en-}|,\kappa_{en-})$ it is fulfilled: $\mu_{en-,1}\equiv\mu_{en-,2}\equiv\mu_{en-,3}=\mu_{en-}$. For example, it is convenient to chose $\mu_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-})=C_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-})$. Accordingly, eqn. (35) could be rewritten as:

$$\boldsymbol{T}_{approx,en-}(c,n,|\alpha|,\bar{r}_0)=C_{en-}(c,n,|\alpha_{en-}|,\kappa_{en-})\ \boldsymbol{U}\ (|\alpha|,\bar{r}_0) \qquad (39)$$

At the point $\alpha_{en-,\min}=-1/\sqrt{2}$, $\bar{\beta}_{en-,\min}=1/\sqrt{2}$, $\min\bar{r}_{0en-}=2L(c,n)$ it is true:

$$\boldsymbol{T}_{approx,en-,\min}(c,n,|\alpha|,\bar{r}_0)=\mu_{en-,\min,i}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min})\boldsymbol{U}\ (|\alpha|,\bar{r}_0), \qquad (40)$$

$$\mathbf{M}_{en-,\min}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min})=(\mu_{en-,\min,i}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min})), \qquad (41)$$

$$\mathbf{M}_{en-,\min}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min})=$$

$$=[\boldsymbol{E}\ \boldsymbol{U}_{en-,\min}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min})]^{-1}\boldsymbol{T}_{en-,\min}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min}), \qquad (42)$$

$$\mathbf{M}_{en-,\min}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min})=$$

$$=\begin{pmatrix} A_{en-,\min}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min})\min\bar{r}_{0en-}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min}) \\ B_{en-,\min}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min})/|\alpha_{en-,\min}| \\ C_{en-,\min}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min}) \end{pmatrix}. \qquad (43)$$

In a simpler form:

$$\mathbf{M}_{en-,\min}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min})=\begin{pmatrix} 2A_{en-,\min}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min})L(c,n) \\ \sqrt{2}B_{en-,\min}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min}) \\ C_{en-,\min}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min}) \end{pmatrix}. \qquad (44)$$

Obviously, it holds $\mu_{en-,\min,1} \equiv \mu_{en-,\min,2} \equiv \mu_{en-,\min,3} = \mu_{en-,\min}$, $\mu_{en-,\min}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min}) = C_{en-,\min}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min})$. Hence, eqn. (40) could become:

$$\boldsymbol{T}_{approx,en-,\min}(c,n,|\alpha|,\bar{r}_0) = C_{en-,\min}(c,n,|\alpha_{en-,\min}|,\kappa_{en-,\min})\,\boldsymbol{U}\,(|\alpha|,\bar{r}_0), \qquad (45)$$

which for normal TE_{01} mode in view of eqns. (28) and (32) is equivalent to:

$$\boldsymbol{T}_{approx,en-,\min}(3,1,|\alpha|,\bar{r}_0) = 2.10363\,71539\,\boldsymbol{U}\,(|\alpha|,\bar{r}_0). \qquad (46)$$

The last relation is accepted as a basis for computation of the approximate values of the matrix $\boldsymbol{T}$ (of its elements A, B and C).

7. NUMERICAL RESULTS FOR THE A, B, C NUMBERS

Tables 6 and 7 juxtapose the results for normal TE_{01} mode, calculated by means of formula (46) and the exact ones obtained, applying the iterative procedure, described in Ref. [4], as a function of $|\alpha|$ and $\bar{r}_0$. For simplicity of the notations in the Tables the subscript "*approx,en–,min*" is shortened to "*approx*". The exact values of numbers, denoted by the subscript "*exact*" are taken from Tables 1-3 [4,34]. The empty places, indicating that for the relevant set ($|\alpha|$, $\bar{r}_0$) there are no values of the A, B, C numbers ($\boldsymbol{T}$ does not exist), are in accordance with criterion (8), or (9). The comparison shows that the approximate values of the elements of matrix deviate insignificantly from the exact ones.

8. APPROXIMATE METHOD FOR DIFFERENTIAL PHASE SHIFT COMPUTATION

Fundamentals of the approach: If the A, B, C numbers are known at certain point $P \in \mathcal{P}$, the differential phase shift, produced at it, for normal TE_{01} mode could be computed from any of the formulae:

$$\Delta\overline{\beta} = \begin{cases} A|\alpha| \\ B/\overline{r}_0 \\ (C/\overline{r}_0)|\alpha| \end{cases}, \tag{47}$$

Table 6. Exact and approximate values of the numbers A, B, C as a function of $\overline{r}_0$ in case $|\alpha| = 0.1$, $c = 3$ and $n = 1$.

$\overline{r}_0$	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0
A_{exact}	0.513 613	0.456 551	0.410 903	0.373 554	0.342 431	0.316 097	0.293 525	0.273 963	0.256 847
A_{approx}	0.525 909	0.467 475	0.420 727	0.382 479	0.350 606	0.323 636	0.300 520	0.280 485	0.262 955
B_{exact}	0.205 445	0.205 448	0.205 451	0.205 455	0.205 459	0.205 463	0.205 467	0.205 472	0.205 477
B_{approx}	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364
C_{exact}	2.054 452	2.054 481	2.054 513	2.054 548	2.054 587	2.054 628	2.054 674	2.054 722	2.054 775
C_{approx}	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637
$\overline{r}_0$	8.5	9.0	9.5	10.0	10.5	11.0	11.5	12.0	12.5
A_{exact}	0.241 745	0.228 321	0.216 311	0.205 502	0.195 722	0.186 833	0.178 716	0.171 276	0.164 432
A_{approx}	0.247 487	0.233 737	0.22 144	0.210 364	0.200 346	0.191 240	0.182 925	0.175 303	0.168 291
B_{exact}	0.205 483	0.205 489	0.205 495	0.205 502	0.205 508	0.205 516	0.205 523	0.205 531	0.205 539
B_{approx}	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364
C_{exact}	2.054 830	2.054 889	2.054 951	2.055 016	2.055 085	2.055 157	2.055 233	2.055 312	2.055 394
C_{approx}	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637
$\overline{r}_0$	13.0	13.5	14.0	14.5	15.0	15.5	16.0	16.5	17.0
A_{exact}	0.158 114	0.152 264	0.146 833	0.141 776	0.137 057	0.132 643	0.128 504	0.124 617	0.120 958
A_{approx}	0.161 818	0.155 825	0.150 260	0.145 078	0.140 242	0.135 719	0.131 477	0.127 493	0.123 743
B_{exact}	0.205 548	0.205 557	0.205 566	0.205 576	0.205 586	0.205 596	0.205 607	0.205 618	0.205 629
B_{approx}	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364
C_{exact}	2.055 480	2.055 569	2.055 662	2.055 758	2.055 857	2.055 960	2.056 066	2.056 175	2.056 288
C_{approx}	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637
$\overline{r}_0$	17.5	18.0	18.5	19.0	19.5	20.0	20.5	21.0	21.5
A_{exact}	0.117 509	0.114 251	0.111 170	0.108 251	0.105 482	0.102 852	0.100 350	0.097 967	0.095 696
A_{approx}	0.120 208	0.116 869	0.113 710	0.110 718	0.107 879	0.105 182	0.102 616	0.100 173	0.097 844
B_{exact}	0.205 640	0.205 652	0.205 665	0.205 677	0.205 690	0.205 704	0.205 717	0.205 731	0.205 746
B_{approx}	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364
C_{exact}	2.056 404	2.056 524	2.056 647	2.056 773	2.056 903	2.057 036	2.057 173	2.057 313	2.057 457
C_{approx}	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637

$$\Delta\bar{\beta} = \begin{cases} AB/C \\ (AC/B)|\alpha|^2 \\ (BC/A)/\bar{r}_0^2 \end{cases}, \tag{48}$$

Table 7. Exact and approximate values of the numbers A, B, C as a function of $|\alpha|$ in case $\bar{r}_0$ = 5 (5) 20, $c = 3$ and $n = 1$.

$\|\alpha\|$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\bar{r}_0$ =5									
A_{exact}	0.410 902	0.411 195	0.411 673	0.412 321	0.413 116	0.414 026			
A_{approx}	0.420 727	0.420 727	0.420 727	0.420 727	0.420 727	0.420 727			
B_{exact}	0.205 451	0.411 195	0.617 510	0.824 642	1.032 790	1.242 078			
B_{approx}	0.210 364	0.420 727	0.631 091	0.841 455	1.051 819	1.262 182			
C_{exact}	2.054 513	2.055 976	2.058 366	2.061 606	2.065 580	2.070 131			
C_{approx}	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637			
$\bar{r}_0$ =10									
A_{exact}	0.205 502	0.205 794	0.206 261	0.206 868	0.207 564	0.208 275	0.208 904	0.209 324	0.209 378
A_{approx}	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364	0.210 364
B_{exact}	0.205 502	0.411 589	0.618 783	0.827 473	1.037 819	1.249 651	1.462 329	1.674 591	1.884 400
B_{approx}	0.210 364	0.420 727	0.631 091	0.841 455	1.051 819	1.262 182	1.472 546	1.682 910	1.893 273
C_{exact}	2.055 016	2.057 944	2.062 612	2.068 682	2.075 638	2.082 752	2.089 041	2.093 238	2.093 778
C_{approx}	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637
$\bar{r}_0$ =15									
A_{exact}	0.137 057	0.137 416	0.137 985	0.138 714	0.139 525				0.140 608
A_{approx}	0.140 242	0.140 242	0.140 242	0.140 242	0.140 242				0.140 242
B_{exact}	0.205 586	0.412 248	0.620 933	0.832 287	1.046 439				1.898 208
B_{approx}	0.210 364	0.420 727	0.631 091	0.841 455	1.051 819				1.893 273
C_{exact}	2.055 857	2.061 242	2.069 777	2.080 717	2.092 877				2.109 121
C_{approx}	2.103 637	2.103 637	2.103 637	2.103 637	2.103 637				2.103 637
$\bar{r}_0$ =20									
A_{exact}	0.102 852	0.103 295	0.104 000						
A_{approx}	0.105 182	0.105 182	0.105 182						
B_{exact}	0.205 704	0.413 180	0.623 998						
B_{approx}	0.210 364	0.420 727	0.631 091						
C_{exact}	2.057 036	2.065 901	2.079 992						
C_{approx}	2.103 637	2.103 637	2.103 637						

$$\Delta\overline{\beta} = ABC\begin{cases} 1/C^2 \\ \left(1/B^2\right)|\alpha|^2 \\ \left(1/A^2\right)/\overline{r}_0^2 \end{cases} \tag{49}$$

whose area of application is specified by the criterion (8) or (9). The first six expressions have been suggested earlier [4,34] whereas the last three ones are original. Introducing the approximate values of corresponding numbers, determined from eqn. (46) in any of these formulae (e.g. the data from Tables 6,7), yields the approximation of $\Delta\overline{\beta}$ for the relevant parameters $|\alpha|$ and $\overline{r}_0$.

Complementary considerations: It is worthwhile to accept the following new notations $T_1 = A$, $T_2 = B$, $T_3 = C$, $U_1 = 1/\overline{r}_0$, $U_2 = |\alpha|$, $U_3 = 1$, $\boldsymbol{T} = (T_i)$ and $\boldsymbol{U} = (U_i)$. Then, eqns. (47)-(49) are rewritten as:

$$\Delta\overline{\beta} = \begin{cases} T_1U_2 \\ T_2U_1 \\ T_3U_1U_2 \end{cases} = \begin{cases} T_1U_2U_3 \\ T_2U_1U_3 \\ T_3U_1U_2 \end{cases}, \tag{50}$$

$$\Delta\overline{\beta} = \begin{cases} T_1T_2/T_3 \\ (T_1T_3/T_2)U_2^2 \\ (T_2T_3/T_1)U_1^2 \end{cases} = \begin{cases} (T_1T_2/T_3)U_3^2 \\ (T_1T_3/T_2)U_2^2 \\ (T_2T_3/T_1)U_1^2 \end{cases}, \tag{51}$$

$$\Delta\overline{\beta} = T_1T_2T_3\begin{cases} 1/T_3^2 \\ \left(1/T_2^2\right)U_2^2 \\ \left(1/T_1^2\right)U_1^2 \end{cases} = T_1T_2T_3\begin{cases} \left(1/T_3^2\right)U_3^2 \\ \left(1/T_2^2\right)U_2^2 \\ \left(1/T_1^2\right)U_1^2 \end{cases}. \tag{52}$$

(In the second variant of the formulae $U_3 = 1$ is introduced for symmetry.) Moreover, for the elements of eqn. (11) it is fulfilled:

$$T_i = \lambda_{ii}U_i. \tag{53}$$

Accordingly, it is easily seen that expressions (50)-(52) are equivalent to:

$$\Delta\overline{\beta} = \lambda \begin{cases} U_1 U_2 \\ U_1 U_2 U_3 \\ U_1 U_2 / U_3 \end{cases} \tag{54}$$

where $\lambda = \lambda_{ii}$ (cf. eqn. (12)). (The last formulae are identical, since $U_3 = 1$). Similarly, it is true:

$$T_{en-,i} = \lambda_{en-,ii} U_{en-,i} . \tag{55}$$

$$T_{en-,\min,i} = \lambda_{en-,\min,ii} U_{en-,\min,i} , \tag{56}$$

in which $\lambda_{en-,ii} = \lambda_{en-}$, $\lambda_{en-,\min,ii} = \lambda_{en-,\min}$,

$$T_{approx,en-,i} = \mu_{en-} U_i , \tag{57}$$

$$T_{approx,en-,\min,i} = \mu_{en-,\min} U_i , \tag{58}$$

$$\Delta\overline{\beta}_{approx,en-} = \mu_{en-} \begin{cases} U_1 U_2 \\ U_1 U_2 U_3 \\ U_1 U_2 / U_3 \end{cases} , \tag{59}$$

$$\Delta\overline{\beta}_{approx,en-,\min} = \mu_{en-,\min} \begin{cases} U_1 U_2 \\ U_1 U_2 U_3 \\ U_1 U_2 / U_3 \end{cases} , \tag{60}$$

$$\Delta\overline{\beta}_{approx,en-} = (C_{en-} / \overline{r}_0)|\alpha| , \tag{61}$$

$$\Delta\overline{\beta}_{approx,en-,\min} = (C_{en-,\min} / \overline{r}_0)|\alpha| . \tag{62}$$

Finally, the formula, suggested to approximately find the differential phase shift, is:

$$\Delta\overline{\beta}_{approx,en-,\min} = (2.1036371539 / \overline{r}_0)|\alpha| . \tag{63}$$

Numerical results for the differential phase shift: Illustrative examples of the computations are given in Tables 8 and 9 for selected parameters $|\alpha|$ and $\overline{r}_0$. The notations, used in the previous Section are preserved here. The exact values of $\Delta\overline{\beta}$ are

Table 8. Exact and approximate values of $\Delta\bar{\beta}$ for normal TE_{01} mode as a function of $\bar{r}_0$ in case $|\alpha|$=0.1.

$\bar{r}_0$	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0
$\Delta\bar{\beta}_{exact}$	0.051 361	0.045 655	0.041 090	0.037 355	0.034 243	0.031 610	0.029 352	0.027 396	0.025 685
$\Delta\bar{\beta}_{approx}$	0.052 591	0.046 747	0.042 073	0.038 248	0.035 061	0.032 364	0.030 052	0.028 048	0.026 295
$\delta\Delta\bar{\beta}$ %	0.1230	0.1092	0.0983	0.0893	0.0818	0.0754	0.0700	0.0652	0.0610
$\bar{r}_0$	8.5	9.0	9.5	10.0	10.5	11.0	11.5	12.0	12.5
$\Delta\bar{\beta}_{exact}$	0.024 174	0.022 832	0.021 631	0.020 550	0.019 572	0.018 683	0.017 871	0.017 128	0.016 443
$\Delta\bar{\beta}_{approx}$	0.024 749	0.023 374	0.022 144	0.021 036	0.020 035	0.019 124	0.018 292	0.017 530	0.016 829
$\delta\Delta\bar{\beta}$ %	0.0575	0.0542	0.0513	0.0486	0.0463	0.0441	0.0421	0.0402	0.0386
$\bar{r}_0$	13.0	13.5	14.0	14.5	15.0	15.5	16.0	16.5	17.0
$\Delta\bar{\beta}_{exact}$	0.015 811	0.015 226	0.014 683	0.014 178	0.013 706	0.013 264	0.012 850	0.012 462	0.012 096
$\Delta\bar{\beta}_{approx}$	0.016 182	0.015 582	0.015 026	0.014 508	0.014 024	0.013 572	0.013 148	0.012 749	0.012 374
$\delta\Delta\bar{\beta}$ %	0.0371	0.0356	0.0343	0.0330	0.0318	0.0308	0.0298	0.0287	0.0278
$\bar{r}_0$	17.5	18.0	18.5	19.0	19.5	20.0	20.5	21.0	21.5
$\Delta\bar{\beta}_{exact}$	0.011 751	0.011 425	0.011 117	0.010 825	0.010 548	0.010 285	0.010 035	0.009 797	0.009 570
$\Delta\bar{\beta}_{approx}$	0.012 021	0.011 687	0.011 371	0.011 072	0.010 788	0.010 518	0.010 262	0.010 017	0.009 784
$\delta\Delta\bar{\beta}$ %	0.0270	0.0262	0.0254	0.0247	0.0240	0.0233	0.0227	0.0220	0.0214

obtained, employing any formula from the sets (47)-(49) with A, B, C numbers, taken from Tables 1-3 [4,34] for $\bar{r}_0 \in [4,10]$ and found by the iterative procedure, described in Ref. [4] for larger $\bar{r}_0$. (The outcomes in Tables 6,7 could be exercised, too.) The quantities $\Delta\bar{\beta}_{approx}$ are specified from eqn. (63). The absolute error $\delta\Delta\bar{\beta}\%=$ $\left|\Delta\bar{\beta}_{exact}-\Delta\bar{\beta}_{approx}\right|$ is less than 1% for the values of parameters chosen. This is an excellent result, in view of the simplicity of the method, compared with the complicated computations, harnessing the sophisticated iterative techniques of the exact one.

9. CONCLUSION

The elements of the theory of the $L(c,n)$ numbers – a new class of real numbers, appearing in the analysis of the purely imaginary zeros of the complex Kummer CHF of specially selected parameters – are presented. They include: the definition,

Table 9. Exact and approximate values of $\Delta\bar{\beta}$ for normal TE_{01} mode as a function of $|\alpha|$ in case $\bar{r}_0$ =5 (5) 20.

$\|\alpha\|$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\bar{r}_0$ =5									
$\Delta\bar{\beta}_{exact}$	0.041 090	0.082 239	0.123 502	0.164 928	0.206 558	0.248 416			
$\Delta\bar{\beta}_{approx}$	0.042 073	0.084 145	0.126 218	0.168 291	0.210 364	0.252 436			
$\delta\Delta\bar{\beta}$ %	0.0983	0.1906	0.2716	0.3363	0.3806	0.4020			
$\bar{r}_0$ =10									
$\Delta\bar{\beta}_{exact}$	0.020 550	0.041 159	0.061 878	0.082 747	0.103 782	0.124 965	0.146 233	0.167 459	0.188 440
$\Delta\bar{\beta}_{approx}$	0.021 036	0.042 073	0.063 109	0.084 145	0.105 182	0.126 218	0.147 255	0.168 291	0.189 327
$\delta\Delta\bar{\beta}$ %	0.0486	0.0914	0.1231	0.1398	0.1400	0.1253	0.1022	0.0832	0.0887
$\bar{r}_0$ =15									
$\Delta\bar{\beta}_{exact}$	0.013 706	0.027 483	0.041 396	0.055 486	0.069 763				0.126 547
$\Delta\bar{\beta}_{approx}$	0.014 024	0.028 048	0.042 073	0.056 097	0.070 121				0.126 218
$\delta\Delta\bar{\beta}$ %	0.0318	0.0565	0.0677	0.0611	0.0358				0.0329
$\bar{r}_0$ =20									
$\Delta\bar{\beta}_{exact}$	0.010 285	0.020 659	0.031 200						
$\Delta\bar{\beta}_{approx}$	0.010 518	0.021 036	0.031 555						
$\delta\Delta\bar{\beta}$ %	0.0233	0.0377	0.0355						

properties, general characteristics and tables, compiled for the first time. Some features of related with them quantities are also object of the investigation.

As a field of application of the numbers, the theory of azimuthally magnetized circular ferrite waveguide phase shifters, working in normal TE_{01} mode, is considered. Definitions of essential for its formulation notions: spaces of phase characteristics and of phase shifter operation, matrix $\boldsymbol{T}$ of the A, B, C numbers and space of the matrices are given. The main point in the paper is the development of an approximate method for computation of the quantities A, B, C. According to it, the first (second) of them is obtained dividing (multiplying) certain real number (denoted by $C_{en-,\min}$ and equal to 2.10363 71539) by the value of the normalized guide radius $\bar{r}_0$ (of the modulus of off-diagonal ferrite tensor element $|\alpha|$), chosen as parameter. The third is accepted to be equal to the number mentioned in the whole area (space) of

phase shifter operation of the structure. A procedure is described, giving $C_{en-,\min}$ in terms of $L(c,n)$ for specific c and n. Using special formulae, involving A, B, C, substantiated earlier, the approximate values of differential phase shift are figured. A discussion reveals also a straightforward way to get the latter just dividing the basic number $C_{en-,\min}$ by $\bar{r}_0$ and subsequently multiplying it by $|\alpha|$. The approach goes on the slight dependence of the first (second) number on $|\alpha|$ ($\bar{r}_0$) and of the third one on both parameters, established numerically. It is very simple and harnesses the iterative schemes, inherent to the exact one, thrifty which is its main advantage. Simultaneously, the error introduced practically could be neglected.

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