GENERAL SPATIAL CORRELATION MODEL IN VON MISES ANGLE OF ARRIVAL DISTRIBUTION ENVIRONMENT

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Abstract
Analytical expressions are derived for the envelope correlation of the signals measured with two closely spaced antennas in an urban multipath environment. The angles-of-arrival (AoA) of the multipath waves are assumed to be von Mises distributed. It are closed-form expressions and therefore easier to implement in for instance MIMO-capacity calculations than previously published formulas that include infinite summations. Moreover, the new expressions are not restricted to an array of isotropic radiators but also hold for an array of dipole antennas.

1. INTRODUCTION
In mobile communication systems multiple waves are propagating between transmitter and receiver. This multipath propagation is caused by electromagnetic wave reflection, diffraction and scattering and transmission through objects like buildings, trees, etc. [1-3]. All these waves arrive at the receiver with different angle-of-arrival (AoA) [4].

For next generation 4G radio systems (using e.g. smart antennas or MIMO) this angular dispersion of radiowaves is becoming increasingly important. In beamforming systems it strongly influences the effect of nulling out interference or directing a beam to obtain a maximum signal level. In MIMO systems the angular dispersion has a major effect on capacity and diversity gain [5-7]. This is because angular dispersion results in a decorrelation of the radio signals at the MIMO antenna elements. A larger angular spread gives a lower signal correlation leading to an increase of MIMO system capacity.

There are two approaches to calculate the signal correlation: 1) based on antenna pattern [8-11]; 2) based on S-parameters [12-13]. In the present paper the first mentioned approach will be used.

If the AoA-distribution is uniform in azimuth the correlation of the envelope signals received at two closely spaced antenna elements is given by a zeroth-order Bessel function of the first kind. This scenario was first considered and analyzed by Clarke [14] and also described in the classic book of Jakes [15]. Because the uniform AoA-distribution does not generally hold, later on various papers appeared that considered other probability density functions (pdf) of AoA, such as Gaussian and von Mises. The von Mises pdf is best suited for AoA because it is a continuous periodical pdf.

In a recent paper by Queiroz et al. expressions were derived for the signal correlation assuming the von Mises pdf [16]. These expressions include infinite summations that must be
truncated in order to obtain numerical results. Furthermore, the radiation patterns of the antenna elements are assumed to be omnidirectional.

In the present paper closed-form analytical expressions of the envelope correlation that only consist of simple Bessel functions are derived and we will extend these easy-to-implement expressions to the case of short dipole antenna elements. These results are novel (to the authors knowledge). Section II deals with the correlation functions in case of small isotropic sources and in section III the obtained analytical results are extended to the case of two similar broadside short dipoles. Numerical results for both cases are given in Section IV. The paper will be concluded with a physical explanation of the obtained results in section V.

2. CALCULATION OF ENVELOPE CORRELATIONS FOR TWO SIMILAR ISOTROPIC ANTENNAS

The geometry of the problem is shown in Fig.1. The probability function (pdf) of the random variable \( \phi \), that is an angle-of-arrival AoA) of a multipath wave, is assumed to be given by the “von Mises model”

\[
p_{\phi}^\kappa (\phi) = \frac{1}{2\pi I_0(\kappa)} e^{\cos(\phi - \phi_p)}
\]  

(1)

where \( \kappa \) is a distribution parameter, \( \phi_p \) is the mean AoA, and \( I_0(\kappa) \) is a modified Bessel function of the first kind and zero order. Small values of the angular spread parameter ( \( \kappa \) ) are related to a large angular spread of the multipath waves, while large values of this parameter are related to a small angular spread around \( \phi_p \).

The pdf (1) satisfies the following normalization condition

\[
\int_{-\pi}^{\pi} p_{\phi}^\kappa (\phi) d\phi = 1
\]  

(2)

The spatial correlation between the signals received with two similar isotropic antennas separated by distance (d) and positioned on the x-axis (Fig.1) is given by the following integral [6]
and here for convenience the relative distance parameter \((\xi = 2\pi d/\lambda)\) is introduced.

The more often used envelope correlation \(\rho_e\) is related with this correlation by the following equation

\[
\rho_e(\xi) = |\rho(\xi)|^2
\]  

\(\text{(4)}\)

Note: in our further investigation the dependence of this functions on other parameters \((\kappa, \phi_p)\) will be assumed but omitted for convenience, and only the dependence on the variable \((\xi)\) will be indicated explicitly.

Now we will make some simple transformations in the exponential term in the integrant of (3). Rewriting this term as

\[
\cos(\kappa \phi_p + j\xi) = S(\xi) \cos(\theta(\xi)),
\]

\(\text{(6)}\)

where after elementary trigonometric transformations

\[
S(\xi) = \sqrt{\kappa^2 - \xi^2 + 2j\kappa \xi \sin \phi_p},
\]

\(\text{(7)}\)

\[
\theta(\xi) = \tan^{-1} \left( \frac{\kappa \sin \phi_p + j\xi}{\kappa \cos \phi_p} \right)
\]

It is easy to be checked that the limiting case \((d \to 0)\) yields to \(S(0) = \kappa, \ \theta(0) = \phi_p\) (an expected result).

Using an integral representation of the modified Bessel function of the second kind and the order zero \(I_0(\cdot)\) \([17]\)

\[
\int_{-\pi}^{\pi} e^{i(\cos\xi)} d\xi = 2\pi I_0(\kappa)
\]

\(\text{(8)}\)

(here \(\phi - \phi_p = \xi\) is assumed) the following result for the spatial correlation function is obtained

\[
\rho_e(\xi) = \frac{I_1(|S(\xi)|)}{I_0(\kappa)}
\]

\(\text{(9)}\)

Then the envelope correlation can be easy obtained by equation (4).

This is a very compact and general result for the correlation function. In Appendix A it is shown that this result is completely equivalent to the result in [16], obtained in terms of
complicated Bessel-trigonometric infinite series. The advantage of our expression (9) in comparison with the one presented in [16] is that ours is much more concise and therefore easier to implement in for instance MIMO-capacity analysis software.

Some numerical results will be shown in section IV of the envelope correlation for a couple of isotropic sources as a function of antenna spacing with the mean AoA ($\phi_p$) and angular spread parameter ($\kappa$) as parameters.

3. CALCULATION OF ENVELOPE CORRELATIONS FOR TWO SIMILAR SHORT DIPOLES

Here only the broadside case is considered, because then the mutual coupling between the antenna elements is smallest, which is needed for MIMO antenna systems. In that case the dipole current is flowing along the array (x-axis), as shown in Fig.2, resulting in an eight shaped radiation pattern in the (x-z) plane with nulls towards adjacent antenna elements. Analysis of the end-fire dipole case can be performed in the same way and in the correlation function the same two special functions (modified Bessel functions of order zero and one) can be introduced.

The general expression for two similar short dipoles at a distance (d) from each other situated in the azimuth plane ($\phi$) is given by the following integral [6]

$$\rho_{\kappa}^{\phi_p}(\xi) = \frac{\int_{-\phi_p}^{\phi_p} e^{i e^{i \kappa \cos(\phi_p - \phi)}} d\phi}{\int_{-\phi_p}^{\phi_p} e^{i e^{i \kappa \cos(\phi_p - \phi)}} d\phi} = \frac{N(\xi)}{D(\kappa)}$$

(10)

where $N(\xi)$ is the numerator, $D(\kappa)$ is the denominator, and $F^2(\phi) = \cos^2\phi$ is the power radiation pattern of the dipole.

The auxiliary function $S(\xi)$ is defined in (7). It is easy to see that the variable in the denominator can be obtained from the numerator by simply letting $\kappa = S(0)$. After using equations (6) and (7) we come to the following expression for the numerator

$$N(\xi) = \int_{-\phi_p}^{\phi_p} \cos^2\phi S(\xi \cos(\phi - \phi_p)) d\phi$$

(11)

After setting $\phi - \phi_p = \sigma$, the integral above can be rewritten in the following form

$$N(\xi) = \int_{-\phi_p}^{\phi_p} \cos^2(\sigma + \phi_p) e^{i e^{i \kappa \cos(\sigma + \phi_p - \phi)}} d\sigma$$

(12)

Figure 2. Two similar parallel short dipoles (broadside case).
and with the substitution $\sigma - \theta = \tau$ the integral becomes

$$N(\xi) = \int_{-\pi}^{\pi} \cos^2 (\tau + \phi_p + \theta) e^{S(\xi) \cos(\theta + \phi_p)} d\tau$$

(13)

and with $\tau + \phi_p = \rho$ the integral can be written as

$$N(\xi) = \int_{-\pi}^{\pi} \cos^2 (\rho + \theta) e^{S(\xi) \cos(\rho)} d\rho$$

(14)

where $\phi_p - \theta = \psi$ is assumed. Next, some transformations are performed in order to, at the end, arrive at closed form expression without integrals. It is well known that

$$\cos(\rho + \theta) = \cos \rho \cos \theta - \sin \rho \sin \theta$$

That yields to

$$N(\xi) = \int_{-\pi}^{\pi} \left[ \cos^2 \rho \cos^2 \theta + \sin^2 \rho \sin^2 \theta + \cos \rho \sin \theta \cos \theta \right] e^{S(\xi) \cos(\rho)} d\rho$$

(16)

In the first integral $\cos^2 \rho = 1 - \sin^2 \rho$ is set and then it is combined with the second integral. Due to symmetry the third integral is equal to zero. The two remaining integrals take the form

$$N(\xi) = \cos^2 \theta \int_{-\pi}^{\pi} (1 - \sin^2 \rho) e^{S(\xi) \cos(\rho)} d\rho + \sin^2 \theta \int_{-\pi}^{\pi} \sin^2 \rho e^{S(\xi) \cos(\rho)} d\rho$$

That can be rearranged as follows

$$N(\xi) = \cos^2 \theta \int_{-\pi}^{\pi} e^{S(\xi) \cos(\rho)} d\rho + (\sin^2 \theta - \cos^2 \theta) \int_{-\pi}^{\pi} \sin^2 \rho e^{S(\xi) \cos(\rho)} d\rho$$

(17)

Here $\psi$ is an arbitrary constant angle, that can be set to zero. Now we recognize in (17) the standard integral [17]

$$\int_{-\pi}^{\pi} \sin^{2m} \rho e^{S(\xi) \cos(\rho)} d\rho = 2\pi \frac{I_{2m}(\kappa)}{\kappa^m} (m = 0, 1)$$

(18)

which result is an extension of (8).

From (10) it is obvious that the denominator is related to the numerator for the particular value: $D(\kappa) = N(S(\xi = 0))$.

Applying (6) and excluding $(\sin \theta, \cos \theta)$ that yields for the numerator the following final expression

$$N(\xi) = \frac{2\pi}{S^2(\xi)} \left[ \kappa^2 \cos^2 \phi_p I_n(S(\xi)) + \left[ S^2(\xi) - 2\kappa^2 \cos^2 \phi_p \right] I_1(S(\xi)) \right]$$

(19)
It is easy now to obtain a suitable expression for the denominator $D(\kappa)$

$$D(\kappa) = 2\pi \left[ \cos^2 \phi_p I_0(\kappa) + \left[ 1 - 2\cos^2 \phi_p \right] \frac{I_1(\kappa)}{\kappa} \right]$$  

(20)

After substitution of the equations (19) and (20) into (10) we come to the final expression of the spatial correlation coefficient $\rho(\xi)$. Applying (4) and taking a square of the absolute value yields the final expression for the envelope correlation $\rho_e(\xi)$.

4. NUMERICAL RESULTS

Here several numerical results of the envelope correlation function for a couple of similar small EM sources are presented. These results will be analysed and discussed in Section V. For comparison reasons, the same numerical values of the parameters $\phi_p$ and $\kappa$ were used to compute the envelope correlations in case of isotropic sources and the broadside short dipoles.

4.1. case ($\phi_p = 0$).

1) isotropic sources

![Graph showing envelope correlation for isotropic sources.](image)
2) broadside dipoles

![Graph of broadside dipoles](image)

(a) ($\kappa = 0.25$)

1) isotropic sources

![Graph of isotropic sources](image)

(b) ($\kappa = 1.0$)
2) broadside dipoles

(b) ($\kappa = 1.0$)

1) isotropic sources

(c) ($\kappa = 4.0$)
2) broadside dipoles

Figure 3. Correlations for the case $\phi_p = 0$

4.2. case ($\phi_p = 45^\circ$)

1) isotropic sources

(a) ($\kappa = 0.25$)
2) broadside dipoles

(a) \( \kappa = 0.25 \)

1) isotropic sources

(b) \( \kappa = 1.0 \)
2) broadside dipoles

(b) ($\kappa = 1.0$)

1) isotropic sources

(c) ($\kappa = 4.0$)
2) broadside dipoles

(c) \( \kappa = 4.0 \)

Figure 4. Correlations for the case \( \phi_p = 45^0 \)

4.3. case \( \phi_p = 90^0 \)

1) isotropic sources

(a) \( \kappa = 0.25 \)
2) broadside dipoles

(a) \( \kappa = 0.25 \)

1) isotropic sources

(b) \( \kappa = 1.0 \)
2) broadside dipoles

1) isotropic sources
5. ANALYSIS AND DISCUSSION

Let’s now try to physically explain the obtained numerical results for the main lobe of the correlation functions shown in Figs. 3-5. Looking at the main lobe for small von Mises parameter kappa (κ = 0.25) we see that the correlation is almost independent of the mean AoA (Figs. 3.a.1, 4.a.1, 5.a.1 and Figs.3.a.2, 4.a.2, 5.a.2). That is because for small kappa the AoA distribution is almost uniform and therefore the array illumination only slightly differs for different mean AoA values.

Generally, the correlation increases with increasing mean AoA (for instance Figs. 3.c.1, 4.c.1, 5.c.1). That can be explained by a decreasing effective (projected) element spacing with increasing mean AoA. For broadside array illumination (mean AoA = 0 degrees) the spatial filtering of the eight shaped radiation pattern of the broadside dipole reduces the effective AoA – spread, leading to a larger correlation in case of the dipole array in comparison with the array of isotropic radiators. This expectation based on physical reasoning agrees with the predictions by our new closed-form expressions (for instance Figs. 3.a.1, 3.a.2). For end-fire array illumination (mean AoA = 90 degrees) and small AoA-spread (κ = 4) the null in the dipole radiation pattern broadens the effective AoA-distribution (smaller effective AoA-spread) and therefore decreases the spatial correlation. Also this expectation is confirmed by the new spatial correlation model presented in this paper (Figs.5.c.1, 5.c.2).

Looking at the sidelobes of the envelope correlation functions (Figs. 3-5) it can be concluded that these are lowest for the dipole antenna configuration for all considered AoA distributions.

It should be noticed that mutual coupling between antenna elements is not included in the present compact analytical expressions. Depending on the antenna type and the distance...
between the array elements an extension of the expressions might be needed. This is the subject of ongoing research.

**APPENDIX A**

Here the *equivalence* of both models will be proved (our and the model presented in [16]). The key point is the so-called “*Addition theorem*” for Bessel functions [18]

\[
Z_n(mR) = \sum_{q=-\infty}^{\infty} Z_{m+q}(mr)J_q(mp)e^{im\theta}
\]

(A.1)

with \( Z_n(\gamma) = J_n(\gamma) \) is arbitrary cylindrical function. Let us set: \( m = 0, \gamma = 1 \). Then, from equation (A.1) we obtain the following formula

\[
J_n(R) = \sum_{q=-\infty}^{\infty} J_q(r)J_q(\rho)e^{im\theta}
\]

(A.2)

where

\[
R = \sqrt{r^2 + \rho^2 - 2r\rho \cos \alpha}
\]

(A.3)

Now let us consider the special case

\[
r = j\xi; \rho = \frac{2\pi d}{\lambda}; \alpha = \phi_p - \frac{\pi}{2}
\]

(A.4)

It can be shown that

\[
J_n(jS) = I_n(S)
\]

(A.7)

This is the relation between the normal and the modified Bessel function of the first kind and zero order. After division with the modified Bessel function \( I_n(\kappa) \) (normalization) we obtain

our expression of the spatial correlation function

\[
\rho^R(\xi) = \frac{I_n(S(\xi))}{I_n(\kappa)}
\]

(A.8)

This equation is the same as our equation (9). By substituting the parameters from (A.4) into the series expansion (A.2) and using the symmetry

\[
J_n(j\xi) = j^n I_n(\kappa)
\]

(A.9)

and the equality

\[
e^{i\theta - \pi/2} = j^{n}e^{i\theta}
\]

(A.10)
we come to the following double infinite series expression for the spatial correlation function

\[
\rho(d/\lambda) = \frac{1}{I_q(\kappa)} \sum_{q=0}^{\infty} I_q(\kappa)J_q(2\pi d/\lambda)e^{i\phi_q} \tag{A.11}
\]

or

\[
\rho(d/\lambda) = \frac{1}{I_q(\kappa)} \sum_{q=0}^{\infty} I_q(\kappa)J_q(2\pi d/\lambda)[\cos(q\phi_q) + j\sin(q\phi_q)] \tag{A.12}
\]

It can be shown that in the real part of this series only even terms \((q = 2n)\) will survive with \((\cos)\) functions of order \((q = 2n)\), while in the imaginary part of the series only odd terms \((q = 2n + 1)\) will survive with \((\sin)\) functions of order \((q = 2n + 1)\). Here the double infinite series \((-\infty < q < \infty)\) can be split into two single infinite series (with positive indices): 1) term with \((q = 0)\); 2) \(2\sum_{q=1}^{\infty} (\bullet)\)

By taking all these considerations into account the following equivalent form of the spatial correlation function can be obtained

\[
\rho(d/\lambda) = J_0(2\pi d/\lambda) + \frac{2}{I_q(\kappa)} \sum_{n=1}^{\infty} I_{2n}(\kappa)J_{2n}(2\pi d/\lambda)\cos(2n\phi_q) + \\
+ j\frac{2}{I_q(\kappa)} \sum_{n=0}^{\infty} I_{2n+1}(\kappa)J_{2n+1}(2\pi d/\lambda)\sin((2n + 1)\phi_q) \tag{A.13}
\]

which was derived in [16] for the spatial correlation function (in the case of adjacent isotropic elements in a linear array).

This derivation proves the equivalence of our simple closed-form result (9) with the more complicated Bessel-trigonometric series expression from [16] (A.13).

REFERENCES


