

Volume 26

Number 1

June 2024

(ISSN 1109-1606)

Journal of
**APPLIED
ELECTROMAGNETISM**

JAE



Institute of Communication and
Computer Systems

Athens - GREECE

Volume 26
Number 1

June 2024
(ISSN 1109-1606)

**JOURNAL
OF
APPLIED ELECTROMAGNETISM**



Institute of Communication and Computer Systems

Athens - GREECE

Volume 26

Number 1

June 2024

**TRANS BLACK SEA REGION UNION OF APPLIED
ELECTROMAGNETISM (BSUAE)**

JOURNAL OF APPLIED ELECTROMAGNETISM

Institute of Communication and Computer Systems

Athens - GREECE

Editor: Panayiotis Frangos (Greece), pfrangos@central.ntua.gr

Honorary Editor: Nikolaos K. Uzunoglu (Greece), nuzu@central.ntua.gr

Board of Associate Editors

D. Dimitrov (Bulgaria), dcd@tu-sofia.bg
V. Dumbrava (Lithuania), vydum@ktu.lt
G. Georgiev (Bulgaria), gngeorgiev@yahoo.com
G. Matsopoulos (Greece), gmatso@esd.ece.ntua.gr

Editorial Board

ALBANIA

G. Bardhyf, bardhylgolemi@live.com
C. Pirro, p_cipo@yahoo.com

ARMENIA

H. Bagdasarian, hovik@seua.sci.am
H. Terzian, hterzian@seua.sci.am

BULGARIA

A. Antonov, asantonov@abv.bg
A. Lazarov, lazarov@bfu.bg
S. Savov, savovsv@yahoo.com

GEORGIA

R. Zaridze, rzaridze@laetsu.org

GERMANY

M. Georgieva – Grosse, mariana.georgieva-grosse@de.bosch.com

GREECE

H. Anastassiu, ANASTASIOU.Christos@haicorp.com
I. Avramopoulos, hav@mail.ntua.gr
G. Fikioris, gfiki@cc.ece.ntua.gr
J. Kanellopoulos, ikanell@cc.ece.ntua.gr
G. Karagiannidis, geokarag@auth.gr
G. Kliros, gskmsa@hol.gr
T. Mathiopoulos, mathio@space.noa.gr
C. Moschovitis, harism@noc.ntua.gr
K. Nikita, knikita@cc.ece.ntua.gr

I. Ouranos, iouranos@central.ntua.gr
E. Papkelis, spapkel@central.ntua.gr
J. Sahalos, sahalos@auth.gr
M. Theologou, theolog@cs.ntua.gr
N. Triantafyllou, nitriant@central.ntua.gr
K. Ksysstra, katksy@central.ntua.gr
A. Malamou, annamalamou@yahoo.gr
S. Bourgiotis, sbourgiotis@mail.ntua.gr

JORDAN

N. Dib, nihad@just.edu.jo

KAZAKSHTAN

S. Sautbekov, sautbek@mail.ru

LITHUANIA

L. Svilainis, linas.svilainis@ktu.lt

RUSSIA

M. Bakunov, bakunov@rf.unn.ru
A. Grigoriev, adgrigoriev@mail.ru

SERBIA

B. Reljin, ereljin@ubbg.etf.bg.ac.yu

SPAIN

E. Gago – Ribas, egr@tsc.uniovi.es
M. Gonzalez – Morales, gonmor@yllera.tel.uva.es

UNITED KINGDOM

G. Goussetis, G.Goussetis@hw.ac.uk

Publishing Department

N. Triantafyllou, nitriant@central.ntua.gr
K. Ksysstra, katksy@central.ntua.gr
A. Malamou, annamalamou@yahoo.gr
S. Bourgiotis, sbourgiotis@mail.ntua.gr

Journal of Applied Electromagnetism

Copyright Form

The undersigned I confirm that I agree the publication of the article

in the Journal of Applied Electromagnetism and the copyright to belong to Trans Black Sea Union of Applied Electromagnetism. I understand that I have the full right to reuse this manuscript for my own purposes.

Name:

Surname:

Address:

E-mail:

Signed:

***Please send the previous form signed either by e-mail to pfrangos@central.ntua.gr , or by fax to the fax number: +30 210 772 2281, attention of Prof. P. Frangos.**

Address

Institute of Communication and Computer Systems,

National Technical University of Athens,

9, Iroon Polytechniou Str.,

157 73 Athens - GREECE

Tel: (+30) 210 772 3694

Fax: (+30) 210 772 2281, attention of Prof. P. Frangos

e-mail: pfrangos@central.ntua.gr

Web site: **<http://jae.ece.ntua.gr>**

**TRANS BLACK SEA REGION UNION OF APPLIED
ELECTROMAGNETISM (BSUAE)**

JOURNAL OF APPLIED ELECTROMAGNETISM (JAE)

Volume 26 Number 1

June 2024

CONTENTS

**ON THE SOLUTION OF CONSTRAINED SYLVESTER-OBSERVER
EQUATION**

Konstadinos H. Kiritsis

1

In this paper explicit necessary and sufficient conditions for the constrained Sylvester-observer equation are established, in order to have a solution over the field of real numbers. Furthermore, a procedure is given for the computation of the solution. Our approach is based on properties of real and polynomial matrices. Applications of the main results of this paper to linear control theory are discussed.

**DESIGN OF FULL ORDER PROPORTIONAL-INTEGRAL OBSERVER FOR
THE STATE ESTIMATION OF DISCRETE-TIME LINEAR TIME-INVARIANT
SYSTEMS**

Konstadinos H. Kiritsis

17

This paper is devoted to the design of full order proportional-integral observer for the state estimation of discrete-time linear time-invariant systems. In particular, explicit necessary and sufficient conditions are established for the existence of proportional-integral observer for the state estimation of discrete-time linear time-invariant systems and a simple procedure is given for the construction of the observer. Our approach is based on properties of real and polynomial matrices.

ON THE SOLUTION OF CONSTRAINED SYLVESTER-OBSERVER EQUATION

Konstadinos H. Kiritsis *

* Hellenic Air Force Academy, Department of Aeronautical Sciences, Division of Automatic Control, Dekelia Air Base, PC 13671 Acharnes, Attikis, Tatoi, Greece

E-mail: konstantinos.kyritsis@hafa.haf.gr

Abstract

In this paper explicit necessary and sufficient conditions for the constrained Sylvester-observer equation are established, in order to have a solution over the field of real numbers. Furthermore, a procedure is given for the computation of the solution. Our approach is based on properties of real and polynomial matrices. Applications of the main results of this paper to linear control theory are discussed.

Keywords: Constrained Sylvester-observer equation, state estimation, reduced order observer.

1. INTRODUCTION

Let \mathbf{R} be the field of real numbers. Furthermore, let \mathbf{A} and \mathbf{C} be given matrices over \mathbf{R} of size $(n \times n)$ and $(p \times n)$ respectively. The problem studied in this paper can be stated as follows: Do there exist, matrices \mathbf{F} , \mathbf{T} and \mathbf{G} over \mathbf{R} of dimensions $(n-p) \times (n-p)$, $(n-p) \times n$ and $((n-p) \times p)$ respectively with \mathbf{F} and \mathbf{T} being Hurwitz stable (i.e., all its eigenvalues have negative real parts), and of full row rank respectively such that

$$\mathbf{TA} - \mathbf{FT} = \mathbf{GC} \quad (1)$$

$$\det \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix} \neq \mathbf{0} \quad (2)$$

If so, give conditions for the existence of matrices \mathbf{T} , \mathbf{F} and \mathbf{G} which satisfy (1) and constraint (2). The equation (1) with constraint (2) is usually called, in bibliography, “constrained Sylvester-observer equation”. This term comes from the fact that equation (1) with constraint (2) arises in the design of reduced order observer for linear time-invariant systems (see Refs. [1] and [2]). Equation (1) with constraint (2) has been studied in the last years from both theoretical and computational point of view. In [3] (see

also [4]) it is proven that if the pair (\mathbf{A}, \mathbf{C}) is observable, then there exist real matrices \mathbf{T} , \mathbf{F} and \mathbf{G} with \mathbf{T} , \mathbf{F} being of full row rank and Hurwitz stable respectively which satisfy (1) and constraint (2). A criterion for choosing the matrix \mathbf{F} which ensures that the solution matrix \mathbf{T} of (1) is of full row rank has been presented in [5]. Furthermore an algorithm is described for the computation of solution. Elegant numerical methods have been developed for the solution of the Sylvester-observer equation in [4, 6-11]. All these methods implicitly construct a full row rank solution of (1) assuming that such a solution exists [5].

As far as we know there are no published explicit necessary and sufficient conditions for the solvability of constrained Sylvester-observer equation. This motivates the present study. In this paper, by using basic concepts and basic results from linear systems and control theory as well as of the theory of matrices, explicit necessary and sufficient conditions for the solvability of constrained Sylvester-observer equation over the field of real numbers are established. In particular, it is proved that constrained Sylvester-observer equation has a solution over the field of real numbers if and only if the pair (\mathbf{A}, \mathbf{C}) is detectable. Furthermore, a procedure is given for the computation of the solution.

2. BASIC CONCEPTS AND PRELIMINARY RESULTS

This section contains lemmas, which are needed to prove the main results of this paper and some basic concepts from linear systems and control theory as well as of the theory of matrices that are used throughout the paper. Let $\mathbf{R}[s]$ be the ring of polynomials with coefficients in \mathbf{R} . Further let C be the field of complex numbers, also let C^+ be the set of all complex numbers λ with $Re(\lambda) \geq 0$. A matrix whose elements are polynomials over $\mathbf{R}[s]$ is termed polynomial matrix. A matrix $\mathbf{U}(s)$ over $\mathbf{R}[s]$ of dimensions $(k \times k)$ is said to be unimodular if and only if its inverse exists and is also polynomial matrix. Every polynomial matrix $\mathbf{W}(s)$ of size $(p \times m)$ with $rank[\mathbf{W}(s)] = r$, can be expressed as [12]

$$\mathbf{U}_1(s) \mathbf{W}(s) \mathbf{U}_2(s) = \mathbf{M}(s) \tag{3}$$

The polynomial matrices $\mathbf{U}_1(s)$ and $\mathbf{U}_2(s)$ are unimodular and the matrix $\mathbf{M}(s)$ is given by

$$\mathbf{M}(s) = \begin{bmatrix} \mathbf{M}_r(s) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{4}$$

The non-singular polynomial matrix $\mathbf{M}_r(s)$ of size $(r \times r)$ in (4) is given by

$$\mathbf{M}_r(s) = \text{diag} [a_1(s), a_2(s), \dots, a_r(s)] \quad (5)$$

The nonzero polynomials $a_i(s)$ for $i=1,2,\dots, r$ are termed invariant polynomials of $\mathbf{W}(s)$ and have the following property

$$a_i(s) \text{ divides } a_{i+1}(s), \text{ for } i=1,2,\dots, r-1 \quad (6)$$

The relationship (3) is called Smith-McMillan form of $\mathbf{W}(s)$ over $\mathbf{R}[s]$. Since the matrices $\mathbf{U}_1(s)$ and $\mathbf{U}_2(s)$ are unimodular and the polynomial matrix $\mathbf{M}_r(s)$ is non-singular, from (3) and (4) it follows that

$$\text{rank}[\mathbf{W}(s)] = \text{rank}[\mathbf{M}_r(s)] = r \quad (7)$$

Let $\mathbf{A}(s)$ and $\mathbf{B}(s)$ be matrices over $\mathbf{R}[s]$ of appropriate dimensions, if there is a matrix $\mathbf{Q}(s)$ over $\mathbf{R}[s]$ of appropriate size such that

$$\mathbf{A}(s) = \mathbf{B}(s)\mathbf{Q}(s) \quad (8)$$

Then the matrix $\mathbf{Q}(s)$ is called a right divisor of the matrix $\mathbf{A}(s)$ and the matrix $\mathbf{A}(s)$ is called a left multiple of the matrix $\mathbf{Q}(s)$ [13].

Let $\mathbf{A}(s)$ and $\mathbf{B}(s)$ be matrices over $\mathbf{R}[s]$ of appropriate dimensions, if there are matrices $\mathbf{D}(s)$, $\mathbf{A}_1(s)$ and $\mathbf{B}_1(s)$ over $\mathbf{R}[s]$ of appropriate dimensions, such that

$$\mathbf{A}(s) = \mathbf{A}_1(s)\mathbf{D}(s), \quad \mathbf{B}(s) = \mathbf{B}_1(s)\mathbf{D}(s) \quad (9)$$

Then the polynomial matrix $\mathbf{D}(s)$ is called common right divisor of polynomial matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$ [13]. A greatest common right divisor $\mathbf{D}(s)$ of two polynomial matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$ is a common right divisor which is a left multiple of every common right divisor of the matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$, [13].

Let \mathbf{A} and \mathbf{C} be matrices over \mathbf{R} of size $(n \times n)$ and $(p \times n)$ respectively and \mathbf{C} not zero. Then there always exists a unimodular matrix $\mathbf{U}(s)$ over $\mathbf{R}[s]$ such that [13]

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{I}_s - \mathbf{A} \end{bmatrix} = \mathbf{U}(s) \begin{bmatrix} \mathbf{V}(s) \\ \mathbf{0} \end{bmatrix} \quad (10)$$

The non-singular polynomial matrix $\mathbf{V}(s)$ of size $(n \times n)$ is a greatest common right divisor of the polynomial matrices \mathbf{C} and $[\mathbf{I}_s - \mathbf{A}]$ [13]. Since the polynomial matrix $\mathbf{U}(s)$ is unimodular from (10) it follows that

$$\text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_s - \mathbf{A} \end{bmatrix} = \text{rank} \begin{bmatrix} \mathbf{V}(s) \\ \mathbf{0} \end{bmatrix} = \text{rank}[\mathbf{V}(s)] = n \quad (11)$$

Definition 1: The nonzero polynomial $c(s)$ over $\mathbf{R}[s]$ is said to be strictly Hurwitz if and only if $c(s) \neq 0, \forall s \in \mathbb{C}^+$.

Definition 2: Let $\mathbf{V}(s)$ be a non-singular matrix over $\mathbf{R}[s]$, of size $(m \times m)$. Also let $c_i(s)$ for $i=1,2, \dots,m$ be the invariant polynomials of polynomial matrix $\mathbf{V}(s)$. The polynomial matrix $\mathbf{V}(s)$ is said to be strictly Hurwitz if and only if the polynomials $c_i(s)$ are strictly Hurwitz for every $i=1,2,\dots,m$, or alternatively, if and only if $\det[\mathbf{V}(s)]$ is a strictly Hurwitz polynomial.

Definition 3: Let \mathbf{A} and \mathbf{C} be matrices over \mathbf{R} of size $(n \times n)$ and $(p \times n)$, respectively. Then the pair (\mathbf{A}, \mathbf{C}) is said to be detectable if and only if there exists a matrix \mathbf{K} over \mathbf{R} of size $(n \times p)$ such the matrix $[\mathbf{A}+\mathbf{K}\mathbf{C}]$ is Hurwitz stable [14].

Definition 4: Let \mathbf{A} and \mathbf{C} be matrices over \mathbf{R} of size $(n \times n)$ and $(p \times n)$, respectively and \mathbf{C} not zero. Then an eigenvalue λ of the matrix \mathbf{A} is said to be observable, if and only if the following condition holds [15]:

$$\text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{I}\lambda - \mathbf{A} \end{bmatrix} = n$$

Let \mathbf{A} be a real matrix of size $(n \times n)$. The spectrum of the matrix \mathbf{A} , is the set of all its eigenvalues and is denoted by $\sigma(\mathbf{A})$. An eigenvalue λ of \mathbf{A} is called a stable eigenvalue if and only if $\text{Re}(\lambda) < 0$. Otherwise, the eigenvalue λ of the matrix \mathbf{A} is said to be unstable.

Lemma 1: Let \mathbf{A} and \mathbf{C} be matrices over \mathbf{R} of size $(n \times n)$ and $(p \times n)$, respectively and \mathbf{C} not zero. Further, let $\sigma(\mathbf{A})$ be the spectrum of the matrix \mathbf{A} . The pair (\mathbf{A}, \mathbf{C}) is detectable, if and only if one of the following equivalent conditions holds [16]:

- (a) $\text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{I}s - \mathbf{A} \end{bmatrix} = n, \forall s \in C^+$
- (b) $\text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{I}\lambda - \mathbf{A} \end{bmatrix} = n, \forall \lambda \in \sigma(\mathbf{A}) \text{ with } \text{Re}(\lambda) \geq 0.$

From condition (b) of Lemma 1 it follows that the pair (\mathbf{A}, \mathbf{C}) is detectable if and only if all unstable eigenvalues of the matrix \mathbf{A} are observable, [16]. The following lemma is taken from [17].

Lemma 2: Let $\mathbf{V}(s)$ be a non-singular polynomial matrix over $\mathbf{R}[s]$, of size $(m \times m)$. Also let $c_i(s)$ for $i=1,2,\dots,m$ be the invariant polynomials of the polynomial matrix $\mathbf{V}(s)$. The polynomial matrix $\mathbf{V}(s)$ is strictly Hurwitz if and only if the following condition holds

- (a) $\text{rank}[\mathbf{V}(s)] = m, \forall s \in C^+$

Proof: Let $\mathbf{V}(s)$ be a strictly Hurwitz polynomial matrix of size $(m \times m)$ with invariant polynomials $c_i(s)$ for $i=1,2,\dots,m$. From Definition 2 it follows that the polynomials $c_i(s)$ are strictly Hurwitz for every $i=1,2,\dots,m$ and therefore from Definition 1 it follows that

$$c_i(s) \neq 0, \forall s \in C^+, \forall i=1,2, \dots, m \quad (12)$$

We define the polynomial matrix

$$\mathbf{V}_m(s) = \text{diag} [c_1(s), c_2(s), \dots, c_m(s)] \quad (13)$$

From (12) and (13) it follows that

$$\text{rank}[\mathbf{V}_m(s)] = \text{rank}\{\text{diag} [c_1(s), c_2(s), \dots, c_m(s)]\} = m, \forall s \in C^+ \quad (14)$$

The Smith-McMillan form of polynomial matrix $\mathbf{V}(s)$ over $\mathbf{R}[s]$ is given by

$$\mathbf{K}(s) \mathbf{V}(s) \mathbf{L}(s) = \mathbf{V}_m(s) \quad (15)$$

where $\mathbf{K}(s)$ and $\mathbf{L}(s)$ are unimodular matrices. Since the matrices $\mathbf{K}(s)$ and $\mathbf{L}(s)$ are unimodular, from (7), (14) and (15) we have that

$$\text{rank}[\mathbf{V}(s)] = m, \forall s \in C^+ \quad (16)$$

This is condition (a) of the Lemma. To prove sufficiency, we assume that condition (a) holds. Using (7) from (13) and (15) we have that

$$\text{rank}[\mathbf{V}(s)] = \text{rank}[\mathbf{V}_m(s)] = \text{rank}\{\text{diag}[c_1(s), c_2(s), \dots, c_m(s)]\} = m \quad (17)$$

Since by assumption condition (a) holds we have that

$$\text{rank}[\mathbf{V}(s)] = m, \forall s \in C^+ \quad (18)$$

Relationships (17) and (18) imply

$$\text{rank}[\mathbf{V}_m(s)] = \text{rank}\{\text{diag} [c_1(s), c_2(s), \dots, c_m(s)]\} = m, \forall s \in C^+ \quad (19)$$

From (19) it follows that

$$c_i(s) \neq 0, \forall s \in C^+, \forall i=1,2, \dots, m \quad (20)$$

Relationship (20) and Definition 1 imply that polynomials $c_i(s)$ are strictly Hurwitz for every $i=1,2,\dots,m$, and therefore according to Definition 2 the non-singular polynomial matrix $\mathbf{V}(s)$ over $\mathbf{R}[s]$, is strictly Hurwitz. This completes the proof.

The following lemma is taken from [17].

Lemma 3: Let \mathbf{A} and \mathbf{C} be matrices over \mathbf{R} matrices of size $(n \times n)$, $(p \times n)$, respectively and \mathbf{C} not zero. Further let $\mathbf{V}(s)$ be a greatest common right divisor of polynomial matrices $[\mathbf{I}s - \mathbf{A}]$ and \mathbf{C} of size $(n \times n)$. The pair (\mathbf{A}, \mathbf{C}) is detectable if and only if the following condition holds:

(a) The polynomial matrix $\mathbf{V}(s)$ is strictly Hurwitz.

Proof: Let the pair (\mathbf{A}, \mathbf{C}) is detectable. Then from condition (a) of Lemma 1 it follows that

$$\text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_S - \mathbf{A} \end{bmatrix} = n, \quad \forall s \in C^+ \quad (21)$$

Since the polynomial matrix $\mathbf{V}(s)$ is the greatest common right divisor of polynomial matrices $[\mathbf{I}_S - \mathbf{A}]$ and \mathbf{C} , from (10) it follows that there exists a unimodular matrix $\mathbf{U}(s)$ such that

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{I}_S - \mathbf{A} \end{bmatrix} = \mathbf{U}(s) \begin{bmatrix} \mathbf{V}(s) \\ \mathbf{0} \end{bmatrix} \quad (22)$$

Since the polynomial matrix $\mathbf{U}(s)$ is unimodular from (11) and (22) it follows that

$$\text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_S - \mathbf{A} \end{bmatrix} = \text{rank} \begin{bmatrix} \mathbf{V}(s) \\ \mathbf{0} \end{bmatrix} = \text{rank}[\mathbf{V}(s)] \quad (23)$$

From relationships (21) and (23) we have that

$$\text{rank}[\mathbf{V}(s)] = n, \quad \forall s \in C^+ \quad (24)$$

Relationship (24) and condition (a) of Lemma 2 imply that the polynomial matrix $\mathbf{V}(s)$ is strictly Hurwitz. This is condition (a) of the Lemma. To prove sufficiency, we assume that the polynomial matrix $\mathbf{V}(s)$ is strictly Hurwitz. Then from Lemma 2 it follows that

$$\text{rank}[\mathbf{V}(s)] = n, \quad \forall s \in C^+ \quad (25)$$

Since the polynomial matrix $\mathbf{V}(s)$ is the greatest common right divisor of polynomial matrices $[\mathbf{I}_S - \mathbf{A}]$ and \mathbf{C} , from (23) and (25) it follows that

$$\text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_S - \mathbf{A} \end{bmatrix} = n, \quad \forall s \in C^+ \quad (26)$$

Condition (a) of Lemma 1 and (26) imply that the pair (\mathbf{A}, \mathbf{C}) is detectable. This completes the proof.

Let \mathbf{A} and \mathbf{C} be matrices over \mathbf{R} matrices of size $(n \times n)$, $(p \times n)$, respectively with $\text{rank}[\mathbf{C}] = p$. Then there exists a non-singular matrix \mathbf{L} over \mathbf{R} of size $(n \times n)$ such that

$$\mathbf{C}\mathbf{L} = \mathbf{C}_1 \quad (27)$$

$$\mathbf{L}^{-1}\mathbf{A}\mathbf{L} = \mathbf{A}_1 \quad (28)$$

The matrices \mathbf{C}_1 and \mathbf{A}_1 are given by

$$\mathbf{C}_1 = [\mathbf{I}_p, \mathbf{0}] \quad (29)$$

$$\mathbf{A}_1 = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \quad (30)$$

where \mathbf{I}_p is the identity matrix of size $(p \times p)$, and \mathbf{A}_{11} , \mathbf{A}_{12} , \mathbf{A}_{21} and \mathbf{A}_{22} are matrices over \mathbf{R} of dimensions $(p \times p)$, $(p \times (n-p))$, $((n-p) \times p)$ and $((n-p) \times (n-p))$ respectively.

Lemma 4: Let \mathbf{A} and \mathbf{C} be matrices over \mathbf{R} matrices of size $(n \times n)$, $(p \times n)$ respectively with $\text{rank}[\mathbf{C}]=p$. The pair (\mathbf{A}, \mathbf{C}) is detectable if and only if the following condition holds:

(a) The pair $(\mathbf{A}_{22}, \mathbf{A}_{12})$ is detectable.

Proof: Let the pair (\mathbf{A}, \mathbf{C}) is detectable. Then from condition (a) of Lemma 1 it follows that

$$\text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_S - \mathbf{A} \end{bmatrix} = n, \quad \forall s \in C^+ \quad (31)$$

From (27), (28), (29) and (30) we have that

$$\mathbf{C} = [\mathbf{I}_p, \mathbf{0}] \mathbf{L}^{-1} = \mathbf{C}_1 \mathbf{L}^{-1}, \quad \mathbf{A} = \mathbf{L} \mathbf{A}_1 \mathbf{L}^{-1} = \mathbf{L} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \mathbf{L}^{-1} \quad (32)$$

Using (32) and after simple algebraic manipulations the (31) can be expressed as follows

$$\begin{aligned} \text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_S - \mathbf{A} \end{bmatrix} &= \text{rank}[\text{diag}[\mathbf{I}_p, \mathbf{L}] \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{I}_S - \mathbf{A}_1 \end{bmatrix} \mathbf{L}^{-1}] = \\ &= \text{rank}[\text{diag}[\mathbf{I}_p, \mathbf{L}] \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{I}_p s - \mathbf{A}_{11} & -\mathbf{A}_{12} \\ -\mathbf{A}_{21} & \mathbf{I}_{n-p} s - \mathbf{A}_{22} \end{bmatrix} \mathbf{L}^{-1}] = n, \quad \forall s \in C^+ \end{aligned} \quad (33)$$

Since the matrices, $\text{diag}[\mathbf{I}_p, \mathbf{L}]$ and \mathbf{L}^{-1} are non-singular, from (33) it follows that

$$\text{rank} \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{I}_p s - \mathbf{A}_{11} & -\mathbf{A}_{12} \\ -\mathbf{A}_{21} & \mathbf{I}_{n-p} s - \mathbf{A}_{22} \end{bmatrix} = n, \quad \forall s \in C^+ \quad (34)$$

Since the n columns of the matrix in the left side of (34) are linearly independent over C , $\forall s \in C^+$, a subset of these columns consisting of the last $(n-p)$ columns must be also linearly independent over C , $\forall s \in C^+$; therefore

$$\text{rank} \begin{bmatrix} \mathbf{0} \\ -\mathbf{A}_{12} \\ \mathbf{I}_{n-p} s - \mathbf{A}_{22} \end{bmatrix} = \text{rank} \begin{bmatrix} -\mathbf{A}_{12} \\ \mathbf{I}_{n-p} s - \mathbf{A}_{22} \end{bmatrix} = \text{rank} \begin{bmatrix} \mathbf{A}_{12} \\ \mathbf{I}_{n-p} s - \mathbf{A}_{22} \end{bmatrix} = (n-p), \quad \forall s \in C^+ \quad (35)$$

Relationship (35) and condition (a) of Lemma 1 imply that the pair $(\mathbf{A}_{22}, \mathbf{A}_{12})$ is detectable. This is condition (a). To prove sufficiency, we assume that the pair $(\mathbf{A}_{22}, \mathbf{A}_{12})$ is detectable. Let $\Gamma(s)$ be a greatest common right divisor of polynomial matrices

$[\mathbf{I}_{n-p}s - \mathbf{A}_{22}]$ and \mathbf{A}_{12} . Then from (10) it follows that there exists a unimodular matrix $\mathbf{U}(s)$ of size $(n \times n)$ such that

$$\begin{bmatrix} -\mathbf{A}_{12} \\ \mathbf{I}_{n-p}s - \mathbf{A}_{22} \end{bmatrix} = \mathbf{U}(s) \begin{bmatrix} \mathbf{\Gamma}(s) \\ \mathbf{0} \end{bmatrix} \quad (36)$$

We define the following matrices

$$\begin{bmatrix} \mathbf{I}_p s - \mathbf{A}_{11} \\ -\mathbf{A}_{21} \end{bmatrix} = \mathbf{U}(s) \begin{bmatrix} \mathbf{E}(s) \\ \mathbf{Z}(s) \end{bmatrix} \quad (37)$$

$$\mathbf{\Delta}(s) = \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{E}(s) & \mathbf{I}_{n-p} \end{bmatrix} \quad (38)$$

$$\mathbf{H}(s) = [\mathbf{Z}(s), \mathbf{0}] \quad (39)$$

$$\mathbf{M}(s) = \text{diag}[\mathbf{I}_p, \mathbf{\Gamma}(s)] \quad (40)$$

where $\mathbf{\Delta}(s)$ is a unimodular matrix of size $(n \times n)$ and $\mathbf{E}(s)$, $\mathbf{Z}(s)$, $\mathbf{H}(s)$ and $\mathbf{M}(s)$ are polynomial matrices of size $((n-p) \times p)$, $(p \times p)$, $(p \times n)$ and $(n \times n)$ respectively. Using (29), (30), (37), (38), (39) and (40) we have that

$$\begin{aligned} \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{I}s - \mathbf{A}_1 \end{bmatrix} &= \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{I}_p s - \mathbf{A}_{11} & -\mathbf{A}_{12} \\ -\mathbf{A}_{21} & \mathbf{I}_{n-p}s - \mathbf{A}_{22} \end{bmatrix} = \text{diag}[\mathbf{I}_p, \mathbf{U}(s)] \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{E}(s) & \mathbf{\Gamma}(s) \\ \mathbf{Z}(s) & \mathbf{0} \end{bmatrix} = \\ &= \text{diag}[\mathbf{I}_p, \mathbf{U}(s)] \text{diag}[\mathbf{\Delta}(s), \mathbf{I}_p] \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-p} \\ \mathbf{Z}(s) & \mathbf{0} \end{bmatrix} \text{diag}[\mathbf{I}_p, \mathbf{\Gamma}(s)] = \\ &= \text{diag}[\mathbf{I}_p, \mathbf{U}(s)] \text{diag}[\mathbf{\Delta}(s), \mathbf{I}_p] \begin{bmatrix} \mathbf{I}_n \\ \mathbf{H}(s) \end{bmatrix} \text{diag}[\mathbf{I}_p, \mathbf{\Gamma}(s)] = \\ &= \text{diag}[\mathbf{I}_p, \mathbf{U}(s)] \text{diag}[\mathbf{\Delta}(s), \mathbf{I}_p] \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{H}(s) & \mathbf{I}_p \end{bmatrix} \begin{bmatrix} \mathbf{I}_n \\ \mathbf{0} \end{bmatrix} \mathbf{M}(s) \end{aligned} \quad (41)$$

Since the matrices $\text{diag}[\mathbf{I}_p, \mathbf{U}(s)]$, $\text{diag}[\mathbf{\Delta}(s), \mathbf{I}_p]$ and $\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{H}(s) & \mathbf{I}_p \end{bmatrix}$ are all unimodular, their product

$$\text{diag}[\mathbf{I}_p, \mathbf{U}(s)] \text{diag}[\mathbf{\Delta}(s), \mathbf{I}_p] \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{H}(s) & \mathbf{I}_p \end{bmatrix}$$

is also a unimodular matrix and therefore from (10) it follows that the matrix $\mathbf{M}(s)$ in (41) is a greatest common right divisor of the polynomial matrices \mathbf{C}_1 , $[\mathbf{I}s - \mathbf{A}_1]$. Detectability of the pair $(\mathbf{A}_{22}, \mathbf{A}_{12})$ and Lemma 3 imply that the matrix $\mathbf{\Gamma}(s)$ in (36) is strictly Hurwitz. From (40) we have that

$$\det[\mathbf{M}(s)] = \det[\mathbf{\Gamma}(s)] \quad (42)$$

Since the matrix $\Gamma(s)$ is strictly Hurwitz, from Definition 2 it follows that the $\det[\Gamma(s)]$ is strictly Hurwitz polynomial and therefore from (42) and Definition 2 it follows that the matrix $\mathbf{M}(s)$ is strictly Hurwitz. Since $\mathbf{M}(s)$ is strictly Hurwitz, from Lemma 3 it follows that the pair $(\mathbf{A}_1, \mathbf{C}_1)$ is detectable. Since the pair $(\mathbf{A}_1, \mathbf{C}_1)$ is detectable and the detectability is invariant under similarity transformation [16], from (32) it follows that the pair (\mathbf{A}, \mathbf{C}) is also detectable. This completes the proof.

Lemma 5. Let \mathbf{A} and \mathbf{C} be matrices over \mathbf{R} of size $(n \times n)$ and $(p \times n)$, respectively. Then the pair (\mathbf{A}, \mathbf{C}) is observable if and only if for every monic polynomial $c(s)$ over $\mathbf{R}[s]$ of degree n there exists a matrix \mathbf{K} over \mathbf{R} of size $(n \times p)$, such that the matrix $[\mathbf{A} + \mathbf{K}\mathbf{C}]$ has characteristic polynomial $c(s)$ [12].

The standard decomposition of unobservable systems given in the following Lemma was first published by Kalman in [18] and can also be found in any standard book of linear systems theory.

Lemma 6: Let \mathbf{A} and \mathbf{C} be matrices over \mathbf{R} of size $(n \times n)$ and $(p \times n)$, respectively. Further, let the pair (\mathbf{A}, \mathbf{C}) is unobservable and \mathbf{C} not zero. Then there exists a non-singular matrix \mathbf{T} of size $(n \times n)$ such that

$$\begin{aligned} \mathbf{T}^{-1}\mathbf{A}\mathbf{T} &= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \\ \mathbf{C}\mathbf{T} &= [\mathbf{C}_1, \mathbf{0}] \end{aligned}$$

The pair $(\mathbf{A}_{11}, \mathbf{C}_1)$ is observable and the eigenvalues of the matrix \mathbf{A}_{22} are the unobservable eigenvalues of the pair (\mathbf{A}, \mathbf{C}) .

Lemma 7: Let \mathbf{A} and \mathbf{C} be matrices over \mathbf{R} of size $(n \times n)$, $(p \times n)$, respectively and \mathbf{C} not zero. Further let

$$\mathbf{A} = \mathbf{T} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \mathbf{T}^{-1}, \quad \mathbf{C} = [\mathbf{C}_1, \mathbf{0}] \mathbf{T}^{-1}$$

with $(\mathbf{A}_{11}, \mathbf{C}_1)$ observable. If the pair (\mathbf{A}, \mathbf{C}) is detectable then the matrix \mathbf{A}_{22} is Hurwitz stable [16].

The following lemma and its proof are based on the results of [12].

Lemma 8: Let \mathbf{A} and \mathbf{C} be matrices over \mathbf{R} of size $(n \times n)$, $(p \times n)$, respectively and \mathbf{C} not zero. Further, let the pair (\mathbf{A}, \mathbf{C}) be detectable. Then there exists a matrix \mathbf{K} over \mathbf{R} of size $(n \times p)$ such that the matrix $[\mathbf{A} + \mathbf{K}\mathbf{C}]$ is Hurwitz stable.

Proof: Let the pair (\mathbf{A}, \mathbf{C}) is detectable. Detectability of the pair (\mathbf{A}, \mathbf{C}) implies that the pair (\mathbf{A}, \mathbf{C}) is either observable or unobservable with stable unobservable eigenvalues. If the pair (\mathbf{A}, \mathbf{C}) is observable, then from Lemma 5 it follows that there exists a matrix \mathbf{K} of appropriate size over \mathbf{R} such that

$$\det[\mathbf{I}s - \mathbf{A} - \mathbf{K}\mathbf{C}] = c(s)$$

where $c(s)$ be an arbitrary monic, strictly Hurwitz polynomial over $\mathbf{R}[s]$ of degree n . Since $c(s)$ is a strictly Hurwitz polynomial over $\mathbf{R}[s]$ from Definition 3 it follows that matrix $[\mathbf{A} + \mathbf{K}\mathbf{C}]$ is Hurwitz stable. Since the notion of observability is a dual of controllability (i.e., observability of the pair (\mathbf{A}, \mathbf{C}) implies controllability of the pair $(\mathbf{A}^T, \mathbf{C}^T)$) [12], the matrix \mathbf{K} can be calculated using known methods for the solution of pole assignment problem by state feedback [12].

If the pair (\mathbf{A}, \mathbf{C}) is unobservable with stable unobservable eigenvalues, then from Lemma 6 and Lemma 7 it follows that there exists a matrix \mathbf{T} such that

$$\mathbf{A} = \mathbf{T} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \mathbf{T}^{-1}, \quad \mathbf{C} = [\mathbf{C}_1, \mathbf{0}] \mathbf{T}^{-1}$$

The pair $(\mathbf{A}_{11}, \mathbf{C}_1)$ is observable and the matrix \mathbf{A}_{22} is Hurwitz stable. Hurwitz stability of the matrix \mathbf{A}_{22} and Definition 3 imply that the polynomial $\chi(s)$ given by

$$\det[\mathbf{I}s - \mathbf{A}_{22}] = \chi(s)$$

is a strictly Hurwitz polynomial. Observability of the pair $(\mathbf{A}_{11}, \mathbf{C}_1)$ and Lemma 5 imply the existence of a matrix \mathbf{K}_1 over \mathbf{R} of appropriate dimensions such that

$$\det[\mathbf{I}s - \mathbf{A}_{11} - \mathbf{K}_1\mathbf{C}_1] = \varphi(s)$$

where $\varphi(s)$ is an arbitrary monic, strictly Hurwitz polynomial over $\mathbf{R}[s]$ of appropriate degree. Since the notion of observability is a dual of controllability (i.e., observability of the pair $(\mathbf{A}_{11}, \mathbf{C}_1)$ implies controllability of the pair $(\mathbf{A}_{11}^T, \mathbf{C}_1^T)$), the matrix \mathbf{K}_1 can be calculated using known methods for the solution of pole assignment problem by state feedback [12]. Putting

$$\mathbf{K} = \mathbf{T} \begin{bmatrix} \mathbf{K}_1 \\ \mathbf{0} \end{bmatrix}$$

we obtain

$$\mathbf{A} + \mathbf{K}\mathbf{C} = \mathbf{T} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \mathbf{T}^{-1} + \mathbf{T} \begin{bmatrix} \mathbf{K}_1 \\ \mathbf{0} \end{bmatrix} [\mathbf{C}_1, \mathbf{0}] \mathbf{T}^{-1} =$$

$$= \mathbf{T} \begin{bmatrix} \mathbf{A}_{11} + \mathbf{K}_1 \mathbf{C}_1 & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \mathbf{T}^{-1}$$

From the last relationships it follows that

$$\det[\mathbf{I}s - \mathbf{A} - \mathbf{K}\mathbf{C}] = \varphi(s)\chi(s)$$

Since the polynomials $\chi(s)$ and $\varphi(s)$ are strictly Hurwitz, the polynomial $\varphi(s)\chi(s)$ is also a strictly Hurwitz polynomial and therefore, from Definition 3, it follows that matrix $[\mathbf{A} + \mathbf{K}\mathbf{C}]$ is Hurwitz stable. This completes the proof.

3. MAIN RESULTS

The theorem that follows is the main result of this paper and gives the necessary and sufficient conditions for the constrained Sylvester-observer equation in order, to have a solution over the field of real numbers. Without any loss of generality in what follows we assume that $\text{rank}[\mathbf{C}] = p$.

Theorem 1. There exists matrices \mathbf{F} , \mathbf{T} and \mathbf{G} with \mathbf{F} , \mathbf{T} being Hurwitz stable and of full row rank respectively which satisfy equation (1) and constraint (2), if and only if the following condition holds:

- (a) The pair (\mathbf{A}, \mathbf{C}) is detectable.

Proof: Let there exists matrices \mathbf{F} , \mathbf{T} and \mathbf{G} with \mathbf{F} , \mathbf{T} being Hurwitz stable and of full row rank respectively which satisfy equation (1) and constraint (2). Also let \mathbf{L} a non-singular matrix over \mathbf{R} of size $(n \times n)$ as in (27), (28), (29) and (30). From (27), (28), (29) and (30) we have that

$$\mathbf{C} = [\mathbf{I}_p, \mathbf{0}] \mathbf{L}^{-1} \quad (43)$$

$$\mathbf{A} = \mathbf{L} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \mathbf{L}^{-1} \quad (44)$$

Let

$$\mathbf{T} = [\mathbf{T}_1, \mathbf{T}_2] \mathbf{L}^{-1} \quad (45)$$

where \mathbf{T}_1 and \mathbf{T}_2 are real matrices of size $((n-p) \times p)$ and $((n-p) \times (n-p))$ respectively.

Using (43) and (45) the matrix $\begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix}$ can be expressed as follows

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{T}_1 & \mathbf{T}_2 \end{bmatrix} \mathbf{L}^{-1} \quad (46)$$

Since according to (2) the matrix $\begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix}$ is non-singular, from (46) it follows that the matrix \mathbf{T}_2 is non-singular; therefore

$$\det[\mathbf{T}_2] \neq 0 \quad (47)$$

Substituting (43), (44) and (45) to (1) we obtain:

$$[\mathbf{T}_1, \mathbf{T}_2] \mathbf{L}^{-1} \mathbf{L} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \mathbf{L}^{-1} - \mathbf{F}[\mathbf{T}_1, \mathbf{T}_2] \mathbf{L}^{-1} = \mathbf{G}[\mathbf{I}_p, \mathbf{0}] \mathbf{L}^{-1} \quad (48)$$

Relationship (48) can be rewritten as follows

$$[\mathbf{T}_1, \mathbf{T}_2] \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} - \mathbf{F}[\mathbf{T}_1, \mathbf{T}_2] = \mathbf{G}[\mathbf{I}_p, \mathbf{0}] \quad (49)$$

From (49) it follows that

$$\mathbf{G} = \mathbf{T}_1 \mathbf{A}_{11} + \mathbf{T}_2 \mathbf{A}_{21} - \mathbf{F} \mathbf{T}_1 \quad (50)$$

$$\mathbf{F} \mathbf{T}_2 = \mathbf{T}_2 \mathbf{A}_{22} + \mathbf{T}_1 \mathbf{A}_{12} \quad (51)$$

Condition (47) implies that the matrix \mathbf{T}_2 is invertible. Using this fact (51) can be rewritten as follows

$$\mathbf{T}_2^{-1} \mathbf{F} \mathbf{T}_2 = \mathbf{A}_{22} + \mathbf{T}_2^{-1} \mathbf{T}_1 \mathbf{A}_{12} \quad (52)$$

Since by assumption the matrix \mathbf{F} is Hurwitz stable, the matrix $\mathbf{T}_2^{-1} \mathbf{F} \mathbf{T}_2$ must be also Hurwitz stable. Hurwitz stability of $\mathbf{T}_2^{-1} \mathbf{F} \mathbf{T}_2$ implies Hurwitz stability of $[\mathbf{A}_{22} + \mathbf{T}_2^{-1} \mathbf{T}_1 \mathbf{A}_{12}]$. Definition 3 and Hurwitz stability of the matrix $[\mathbf{A}_{22} + \mathbf{T}_2^{-1} \mathbf{T}_1 \mathbf{A}_{12}]$ imply detectability of the pair $(\mathbf{A}_{22}, \mathbf{A}_{12})$. Detectability of the pair $(\mathbf{A}_{22}, \mathbf{A}_{12})$ and Lemma 4 imply detectability of the pair (\mathbf{A}, \mathbf{C}) . This is condition (a) of the Theorem. To prove sufficiency, we assume that condition (a) holds. Detectability of the pair (\mathbf{A}, \mathbf{C}) and Lemma 4 imply detectability of the pair $(\mathbf{A}_{22}, \mathbf{A}_{12})$. Since the the pair $(\mathbf{A}_{22}, \mathbf{A}_{12})$ is detectable, from Lemma 8 it follows that there exists a real matrix \mathbf{K} of appropriate dimensions such that the matrix $[\mathbf{A}_{22} + \mathbf{K} \mathbf{A}_{12}]$ is Hurwitz stable, that is

$$\det[\mathbf{I}_{n-p} s - \mathbf{A}_{22} - \mathbf{K} \mathbf{A}_{12}] = c(s) \quad (53)$$

where $c(s)$ is a strictly Hurwitz polynomial over $\mathbf{R}[s]$ of degree $(n-p)$. The matrix \mathbf{K} can be calculated as in the proof of Lemma 8. Let

$$\mathbf{T} = [\mathbf{T}_1, \mathbf{T}_2] \mathbf{L}^{-1} = [\mathbf{K}, \mathbf{I}_{n-p}] \mathbf{L}^{-1} \quad (54)$$

$$\mathbf{F} = [\mathbf{A}_{22} + \mathbf{K} \mathbf{A}_{12}] \quad (55)$$

$$\mathbf{G} = \mathbf{K} \mathbf{A}_{11} + \mathbf{A}_{21} - [\mathbf{A}_{22} + \mathbf{K} \mathbf{A}_{12}] \mathbf{K} \quad (56)$$

Using (54), (55), (56), (43) and (44) we have that

$$\begin{aligned}
\mathbf{TA} - \mathbf{FT} &= [\mathbf{K}, \mathbf{I}_{n-p}] \mathbf{L}^{-1} \mathbf{A} - [\mathbf{A}_{22} + \mathbf{KA}_{12}] [\mathbf{K}, \mathbf{I}_{n-p}] \mathbf{L}^{-1} = \\
&= \{[\mathbf{K}, \mathbf{I}_{n-p}] \mathbf{L}^{-1} \mathbf{L} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \mathbf{L}^{-1} - [\mathbf{A}_{22} + \mathbf{KA}_{12}] [\mathbf{K}, \mathbf{I}_{n-p}] \} \mathbf{L}^{-1} = \\
&= \{[\mathbf{K}, \mathbf{I}_{n-p}] \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} - [\mathbf{A}_{22} + \mathbf{KA}_{12}] [\mathbf{K}, \mathbf{I}_{n-p}] \} \mathbf{L}^{-1} = \\
&= \{[(\mathbf{KA}_{11} + \mathbf{A}_{21}), (\mathbf{A}_{22} + \mathbf{KA}_{12})] - [(\mathbf{A}_{22} + \mathbf{KA}_{12})\mathbf{K}, (\mathbf{A}_{22} + \mathbf{KA}_{12})]\} \mathbf{L}^{-1} = \\
&= \{[(\mathbf{KA}_{11} + \mathbf{A}_{21}) - (\mathbf{A}_{22} + \mathbf{KA}_{12})\mathbf{K}], \mathbf{0}\} \mathbf{L}^{-1} = \\
&= \{[(\mathbf{KA}_{11} + \mathbf{A}_{21}) - (\mathbf{A}_{22} + \mathbf{KA}_{12})\mathbf{K}][\mathbf{I}_p, \mathbf{0}]\} \mathbf{L}^{-1} = \mathbf{GC} \quad (57)
\end{aligned}$$

Using (43) and (54) we have that

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{K} & \mathbf{I}_{n-p} \end{bmatrix} \mathbf{L}^{-1} \quad (58)$$

From (54) it follows that the matrix \mathbf{T} is of full row rank and from (53) and (55) we have that the matrix \mathbf{F} is Hurwitz stable. Relationships (57) and (58) imply that the matrices \mathbf{T} , \mathbf{F} and \mathbf{G} given by (54), (55) and (56) satisfy equation (1) and constraint (2). This completes the proof.

The sufficiency part of the proof of Theorem 1 provides a construction of the matrices \mathbf{T} , \mathbf{F} and \mathbf{G} which satisfy equation (1) and constrained (2). The major steps of this construction are given below.

CONSTRUCTION

Given: \mathbf{A} and \mathbf{C} with $\text{rank}[\mathbf{C}] = p$.

Find: \mathbf{T} , \mathbf{F} and \mathbf{G}

Step 1: Check condition (a) of Theorem 1. If this condition is satisfied go to *Step 2*. If condition (a) is not satisfied, the solution of equation (1) with constrained (2) is impossible.

Step 2: Find a non-singular matrix \mathbf{L} over \mathbf{R} of dimensions $(n \times n)$ such that

$$\mathbf{CL} = [\mathbf{I}_p, \mathbf{0}]$$

Step 3: Calculate the following real matrix

$$\mathbf{L}^{-1} \mathbf{A} \mathbf{L} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix},$$

Step 4: Detectability of the pair (\mathbf{A}, \mathbf{C}) and Lemma 4 imply detectability of the pair $(\mathbf{A}_{22}, \mathbf{A}_{12})$. Since the the pair $(\mathbf{A}_{22}, \mathbf{A}_{12})$ is detectable from Lemma 8 it follows that there exists

a real matrix \mathbf{K} of appropriate size such that the matrix

$$[\mathbf{A}_{22} + \mathbf{K}\mathbf{A}_{12}]$$

is Hurwitz stable. The matrix \mathbf{K} can be calculated as in the proof of Lemma 8.

Step 5: Put

$$\mathbf{T} = [\mathbf{K}, \mathbf{I}_{n-p}] \mathbf{L}^{-1}$$

$$\mathbf{F} = [\mathbf{A}_{22} + \mathbf{K}\mathbf{A}_{12}]$$

$$\mathbf{G} = \mathbf{K}\mathbf{A}_{11} + \mathbf{A}_{21} - [\mathbf{A}_{22} + \mathbf{K}\mathbf{A}_{12}]\mathbf{K}$$

4. APPLICATION TO CONTROL THEORY

Consider a linear time-invariant system described by the following state-space equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (59a)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \quad (59b)$$

where \mathbf{A} , \mathbf{B} and \mathbf{C} are real matrices of size $(n \times n)$, $(n \times m)$ and $(p \times n)$ respectively, $\mathbf{x}(t)$ is the state vector of dimensions $(n \times 1)$, $\mathbf{u}(t)$ is the vector of inputs of dimensions $(m \times 1)$ and $\mathbf{y}(t)$ the vector of outputs of dimensions $(p \times 1)$. Consider also a linear time-invariant system described by the following state-space equations [1]

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t) + \mathbf{G}\mathbf{y}(t) + \mathbf{P}\mathbf{u}(t) \quad (60)$$

where \mathbf{F} , \mathbf{G} and \mathbf{P} are real matrices of size $((n-p) \times (n-p))$, $((n-p) \times p)$ and $((n-p) \times m)$ respectively and $\mathbf{z}(t)$ is the state vector of dimensions $((n-p) \times 1)$.

Theorem 2. The system (60) is an observer of order $(n-p)$ of the system described by equations (59a) and (59b) if the following condition holds:

- (a) The pair (\mathbf{A}, \mathbf{C}) is detectable.

Proof: Detectability of the pair (\mathbf{A}, \mathbf{C}) and Theorem 1 imply the existence of a Hurwitz stable matrix \mathbf{F} , of full row rank matrix \mathbf{T} and the matrix \mathbf{G} of dimensions $((n-p) \times (n-p))$, $((n-p) \times n)$ and $((n-p) \times p)$ respectively such that

$$\mathbf{T}\mathbf{A} - \mathbf{F}\mathbf{T} = \mathbf{G}\mathbf{C} \quad (61)$$

$$\det \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix} \neq 0 \quad (62)$$

Let the state estimation error be defined as follows [1]

$$\mathbf{e}(t) = \mathbf{z}(t) - \mathbf{T}\mathbf{x}(t) \quad (63)$$

Then by taking the derivative of (63) and using (59a) and (60) we have that

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{z}}(t) - \mathbf{T}\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{z}(t) + \mathbf{G}\mathbf{y}(t) + \mathbf{P}\mathbf{u}(t) - \mathbf{T}[\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)] \quad (64)$$

Substituting (59b) while adding and subtracting $\mathbf{F}\mathbf{T}\mathbf{x}(t)$ in (64) [1], we have that

$$\dot{\mathbf{e}}(t) = \mathbf{F}\mathbf{e}(t) - [\mathbf{T}\mathbf{A} - \mathbf{F}\mathbf{T} - \mathbf{G}\mathbf{C}]\mathbf{x}(t) + [\mathbf{P} - \mathbf{T}\mathbf{B}]\mathbf{u}(t) \quad (65)$$

Let

$$\mathbf{P} = \mathbf{T}\mathbf{B} \quad (66)$$

Using (66) and (61), from (65) we have that

$$\dot{\mathbf{e}}(t) = \mathbf{F}\mathbf{e}(t) \quad (67)$$

Since the matrix \mathbf{F} is Hurwitz stable, from (67) we have that

$$\lim_{t \rightarrow +\infty} \mathbf{e}(t) = \mathbf{0} \quad (68)$$

for any $\mathbf{x}(0)$, $\mathbf{z}(0)$, $\mathbf{e}(0)$ and $\mathbf{u}(t)$. Hence, from (63) and (68) it follows that $\mathbf{z}(t)$ is an estimate of $\mathbf{T}\mathbf{x}(t)$ and therefore the system (60) is an observer of order $(n-p)$ of the system described by equations (59a) and (59b) [1]. The observer matrices \mathbf{F} , \mathbf{G} and \mathbf{P} can be calculated by solving equation (61) and using relationship (66). Taking into account eqns. (62), (63), (68) and (59b), one can always find an estimate $\hat{\mathbf{x}}(t)$ of $\mathbf{x}(t)$ of the system described by equations (59a) and (59b) as follows [1]

$$\hat{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{C} \\ \mathbf{T} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y}(t) \\ \mathbf{z}(t) \end{bmatrix} \quad (69)$$

This completes the proof.

5. CONCLUSIONS

In this paper explicit necessary and sufficient conditions for the constrained Sylvester-observer equation in order, to have a real solution, are established. The proof of the main results of this paper is constructive and furnishes a procedure for the computation of solution. The results of this paper are useful in studying the problems of linear control theory that can be converted to the solution of constrained Sylvester-observer equation. A typical example is the problem of the existence and construction of reduced order observer for the state estimation of continuous-time linear time invariant systems.

REFERENCES

- [1] B.N. Datta, Numerical Methods for Linear Control Systems Design and Analysis, Elsevier, Academic Press, 2004.
- [2] V.L. Syrmos and F.L. Lewis, “Coupled and constrained Sylvester equations in system design”, Circuits, Systems and Signal Processing, vol.13, no. 6, pp. 663-694, 1994.
- [3] D.G.Luenberger, “An Introduction to Observers”, IEEE Trans. Autom. Control vol.16 no. 6, pp.596-602, 1971.
- [4] P. Van Dooren, “Reduced order observers: a new algorithm and proof”, Systems Contr. Letters, vol.4, no.5, pp. 243–251, 1984.
- [5] K. Datta and M.Thapa, “Unique full-rank solution of the Sylvester-observer equation and its application to state estimation in control design”, Numerical Linear Algebra in Signals, Systems and Control, Springer, Dordrecht, pp. 185-200, 2011.
- [6] C. Bischof, B. N. Datta and A. Purkayastha, “A parallel algorithm for the Sylvester - observer equation”, SIAM J. Scient. Computing, vol.17, no. 3, pp. 686- 698, 1996.
- [7] D. Calvetti, F. L. Lewis and L. Reichel, “On the solution of large Sylvester-observer equation”, Numerical Linear Algebra Applications, vol. 8, no. 6, pp. 435–451, 2001.
- [8] J. Carvalho and B.N. Datta, “A new block algorithm for the Sylvester-observer equation arising in state-estimation”, Proceedings of the IEEE International Conference on Decision and Control, pp. 3398–3403, 2001.
- [9] J. Carvalho, K. Datta and Y. G. Hong, “A new block algorithm for full-rank solution of the Sylvester observer equation” IEEE Trans. Automatic Contr, vol.48, no.12, pp. 2223-2228, 2003
- [10] B.N. Datta and Y. Saad, “Arnoldi Methods for large Sylvester-like observer matrix equations and an associated algorithm for partial spectrum assignment”, LinearAlgebra and its Applications, vol. 154-156, pp. 225-244, 1991.
- [11] [B. N. Datta and D. Sarkissian, “Block algorithms for state estimation and functional observers”, Proceedings of the IEEE International Symposium on Computer-Aided Control System Design, pp. 19–23, 2000.
- [12] V. Kucera, Analysis and Design of Discrete Linear Control Systems, Prentice Hall, London, 1991.
- [13] W.A. Wolowich, Linear Multivariable Systems, Springer Verlag, Berlin, 1974.
- [14] W.M. Wonham, “On a matrix Riccati equation of stochastic control”, SIAM Journal of Control, vol. 6, no. 4, pp. 681-697, 1968.
- [15] H. L. Tredelman, A.A Stoorvogel and M. Hautus, Control theory for linear systems Springer-Verlag, London, 2001.
- [16] K. Zhou, J.C. Doyle and K. Glover, Robust and Optimal Control, Prentice-Hall, New Jersey, 1996.
- [17] K. H. Kiritsis, “On the existence of Luenberger reduced order observer”, accepted for publication in Control and Cybernetics, 2023
- [18] R.E. Kalman, “Mathematical description of linear dynamical systems”, SIAM Journal Control and Optimization, vol. 1, no. 2, pp. 152-192, 1963

DESIGN OF FULL ORDER PROPORTIONAL-INTEGRAL OBSERVER FOR THE STATE ESTIMATION OF DISCRETE-TIME LINEAR TIME-INVARIANT SYSTEMS

Konstadinos H. Kiritsis *

* Hellenic Air Force Academy, Department of Aeronautical Sciences, Division of Automatic Control, Dekelia Air Base, PC 13671 Acharnes, Attikis, Tatoi, Greece

E-mail: konstantinos.kyritsis@hafa.haf.gr

Abstract

This paper is devoted to the design of full order proportional-integral observer for the state estimation of discrete-time linear time-invariant systems. In particular, explicit necessary and sufficient conditions are established for the existence of proportional-integral observer for the state estimation of discrete-time linear time-invariant systems and a simple procedure is given for the construction of the observer. Our approach is based on properties of real and polynomial matrices.

Keywords: Proportional-integral observer, necessary and sufficient conditions, discrete-time linear time-invariant systems.

1. INTRODUCTION

In 1971 Luenberger proposed the full order observer for the state estimation of linear time-invariant systems [1]. In [2] Wojciechowski added an additional term to Luenberger's full order observer for the state estimation of single-input single-output linear time-invariant systems. This term is proportional to the integral of the output estimation error. The resulting new observer was called proportional-integral observer and has a long and rich history. The main results of [2] were generalized to linear multivariable time-varying systems in [3], in particular in [3] implicit necessary and sufficient conditions for the existence of proportional-integral observer for the state estimation of linear multivariable time-varying systems have been established. In [4] a reduced order proportional-integral observer for the state estimation of linear multivariable time-varying systems was first considered. In [5] the robustness property of feedback control systems using a proportional-integral observer was studied. In [6]

necessary and sufficient conditions have been derived under which the proportional-integral observer achieves Exact Loop Transfer Recovery for continuous-time linear time-invariant systems. Similar results have been obtained in [7] for discrete-time linear time-invariant systems. In [8] it was proved that the proportional-integral observer can estimate the state not only of linear time-invariant systems but also of systems with arbitrary external input which appear as unknown input, nonlinearity or unmodeled dynamics.

In [9] it was shown that, for some classes of systems the proportional-integral observer has the ability to completely decouple the modeling uncertainties while keeping satisfactory convergence properties. Furthermore a comparison of classical proportional observer to proportional-integral observer was given using a simulation example. A parametric eigenstructure assignment design approach for proportional-integral observers for the state estimation of continuous-and discrete-time linear time-invariant systems was proposed in [10] and [11] respectively. In [12] a proportional-integral observer based sliding mode controller was proposed for nonlinear hydraulic differential cylinder systems affected by uncertainties. In [13] an optimization method based on a genetic algorithm for the computation of gains of proportional-integral observer for the estimation of state variables of an induction motor is presented. Proportional-integral observer-based approaches for fault detection were developed in [14-16] and references given therein. The proportional-integral observer literature is extremely rich; for more complete references, we refer the reader to [17], [18] and [19]. To the best of our knowledge the problem of design of full order proportional-integral observer for the state estimation of discrete-time linear time-invariant systems, is still an open problem. This motivates the present study. Associated with the design of full order proportional-integral observer for the state estimation of discrete-time linear time-invariant systems are two fundamental questions, i.e. the question of solvability and the question of computability. A major effort in solvability is to determine necessary and sufficient conditions for the existence of a full order proportional-integral observer. The main concern associated with computability, on the other hand, is to develop a procedure for the construction of the proportional-integral observer.

In this paper, these questions have been completely answered. In particular, by using basic concepts and basic results from linear systems and control theory as well as of the theory of matrices are established explicit necessary and sufficient conditions for the existence of a full order proportional-integral observer for the state estimation of discrete-time linear time-invariant systems and a simple procedure is given for the construction of the proportional-integral observer.

2. PROBLEM STATEMENT

Consider a discrete-time linear time-invariant system described by the following state-space equations

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \quad (1)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) \quad (2)$$

where \mathbf{A} , \mathbf{B} and \mathbf{C} are real matrices of size $(n \times n)$, $(n \times m)$ and $(p \times n)$ respectively, $\mathbf{x}(k)$ is the state vector of size $(n \times 1)$, $\mathbf{u}(k)$ is the vector of inputs of size $(m \times 1)$ and $\mathbf{y}(k)$ is the vector of outputs of size $(p \times 1)$. In what follows without any loss of generality we assume that

$$\text{rank}[\mathbf{C}] = p \quad (3)$$

Let us consider a discrete-time linear time-invariant system described by the equations

$$\hat{\mathbf{x}}(k+1) = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(k) + \mathbf{L}\mathbf{y}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{F}\mathbf{v}(k) \quad (4)$$

$$\mathbf{v}(k+1) = \mathbf{v}(k) + [\mathbf{y}(k) - \mathbf{C}\hat{\mathbf{x}}(k)] \quad (5)$$

where $\hat{\mathbf{x}}(k)$ is the state vector of dimensions $(n \times 1)$, $\mathbf{v}(k)$ is a vector of size $(p \times 1)$ and \mathbf{L} and \mathbf{F} are real matrices of size $(n \times p)$ respectively. The discrete-time linear time-invariant system described by the equations(4) and (5) is a proportional-integral observer of order n for the system described by the equations (1) and (2), if and only if for arbitrary initial conditions $\hat{\mathbf{x}}(0)$, $\mathbf{x}(0)$ and any input $\mathbf{u}(k)$, the following relationships hold [11]

$$\lim_{k \rightarrow +\infty} \mathbf{e}(k) = \mathbf{0} \quad (6)$$

$$\lim_{k \rightarrow +\infty} \mathbf{v}(k) = \mathbf{0} \quad (7)$$

where $\mathbf{e}(k) = [\hat{\mathbf{x}}(k) - \mathbf{x}(k)]$ is the state estimation error, $\hat{\mathbf{x}}(k)$ is an estimate of the state vector $\mathbf{x}(k)$ and $\mathbf{v}(k)$ is a vector representing the integral of the weighted output estimation error [11]. The relationships (6) and (7) are simultaneously satisfied if and only if the matrix

$$\begin{bmatrix} \mathbf{A} - \mathbf{L}\mathbf{C} & \mathbf{F} \\ -\mathbf{C} & \mathbf{I}_p \end{bmatrix} \quad (8)$$

of size $((n+p) \times (n+p))$ is Schur stable, i.e. all its eigenvalues have magnitude less than 1 [11]. Thus the problem of the design of the proportional-integral observer of order n can be stated as follows : Do there exist real matrices \mathbf{L} and \mathbf{F} of appropriate dimensions such that the matrix given by (8) is Schur stable? If so, give conditions for existence and a procedure for the calculation of the real matrices \mathbf{L} and \mathbf{F} .

3. BASIC CONCEPTS AND PRELIMINARY RESULTS

This section contains lemmas, which are needed to prove the main results of this paper and some basic notions from linear systems and control theory as well as of the theory of matrices that are used throughout the paper.

Let \mathcal{R} be the field of real numbers. Also let $\mathcal{R}[z]$ be the ring of polynomials with coefficients in \mathcal{R} . Further, let C be the field of complex numbers, also let C^+ be the set of all complex numbers λ with $|\lambda| \geq 1$. The units of $\mathcal{R}[z]$ are polynomials of zero degree, i.e. all nonzero finite real numbers. A polynomial over $\mathcal{R}[z]$ is said to be non-unit if and only if it has nonzero degree. A matrix whose elements are polynomials over $\mathcal{R}[z]$ is termed a polynomial matrix. A polynomial matrix $\mathbf{U}(z)$ over $\mathcal{R}[z]$ of size $(q \times q)$ whose determinant is a unit of $\mathcal{R}[z]$ is termed unimodular matrix [20]. Every polynomial matrix $\mathbf{M}(z)$ of size $(m \times p)$ with $rank[\mathbf{M}(z)]=r$, can be expressed as [20]

$$\mathbf{U}_1(z) \mathbf{M}(z) \mathbf{U}_2(z) = \begin{bmatrix} \mathbf{M}_r(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (9)$$

The polynomial matrices $\mathbf{U}_1(z)$ and $\mathbf{U}_2(z)$ are unimodular and the non-singular polynomial matrix $\mathbf{M}_r(z)$ of size $(r \times r)$ in (9) is given by

$$\mathbf{M}_r(z) = \text{diag} [a_1(z), a_2(z), \dots, a_r(z)] \quad (10)$$

The nonzero polynomials $a_i(z)$ for $i=1,2,\dots, r$ are termed invariant polynomials of $\mathbf{M}(z)$ and have the following property

$$a_i(z) \text{ divides } a_{i+1}(z), \text{ for } i=1, 2, \dots, r-1 \quad (11)$$

The relationship (9) with $\mathbf{M}_r(z)$ given by (10) is called Smith-McMillan form of $\mathbf{M}(z)$ over $\mathcal{R}[z]$. Since the matrices $\mathbf{U}_1(z)$ and $\mathbf{U}_2(z)$ are unimodular and the polynomial matrix $\mathbf{M}_r(z)$ given by (10) is non-singular, from (9) and (10) it follows that

$$\text{rank}[\mathbf{M}(z)] = \text{rank}[\mathbf{M}_r(z)] = r \quad (12)$$

Definition 1: The nonzero polynomial $c(z)$ over $\mathcal{R}[z]$ is said to be strictly Schur if and only if $c(z) \neq 0, \forall z \in C^+$.

Definition 2: Matrix \mathbf{A} over \mathcal{R} of size $(n \times n)$ is said to be Schur stable if and only if all eigenvalues of the matrix \mathbf{A} have magnitude less than 1, or alternatively if and only if the characteristic polynomial of the matrix \mathbf{A} is a strictly Schur polynomial.

Definition 3: Let \mathbf{A} and \mathbf{C} be matrices over \mathcal{R} of size $(n \times n)$ and $(p \times n)$, respectively. Then the pair (\mathbf{A}, \mathbf{C}) is said to be detectable if and only if there exists a matrix \mathbf{K} over \mathcal{R} of size $(n \times p)$ such that the matrix $[\mathbf{A} + \mathbf{K}\mathbf{C}]$ is Schur stable [21].

Definition 4: Let \mathbf{A} and \mathbf{C} be matrices over \mathcal{R} of size $(n \times n)$ and $(p \times n)$, respectively and \mathbf{C} not zero. Then an eigenvalue λ of the matrix \mathbf{A} is said to be observable, if and only if the following condition holds [22]:

$$\text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n \lambda - \mathbf{A} \end{bmatrix} = n$$

Let \mathbf{A} be a real matrix of size $(n \times n)$. The spectrum of the matrix \mathbf{A} , is the set of all its eigenvalues and is denoted by $\sigma(\mathbf{A})$. An eigenvalue λ of \mathbf{A} is called a stable eigenvalue if and only if $|\lambda| < 1$. The eigenvalue λ of the matrix \mathbf{A} is said to be unstable if and only if $|\lambda| \geq 1$.

Lemma 1: Let \mathbf{A} and \mathbf{C} be matrices over \mathcal{R} of size $(n \times n)$ and $(p \times n)$, respectively and \mathbf{C} not zero. Further let $\sigma(\mathbf{A})$ be the spectrum of the matrix \mathbf{A} . The pair (\mathbf{A}, \mathbf{C}) is detectable if and only if one of the following equivalent conditions holds [23] :

$$(a) \text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n z - \mathbf{A} \end{bmatrix} = n, \forall z \in C^+$$

$$(b) \text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n \lambda - \mathbf{A} \end{bmatrix} = n, \forall \lambda \in \sigma(\mathbf{A}) \text{ with } |\lambda| \geq 1$$

From condition (b) of Lemma 1 it follows that the pair (\mathbf{A}, \mathbf{C}) is detectable if and only if all unstable eigenvalues of the matrix \mathbf{A} are observable [23].

Lemma 2. Let \mathbf{A} and \mathbf{C} be matrices over \mathcal{R} of size $(n \times n)$ and $(p \times n)$, respectively. Then the pair (\mathbf{A}, \mathbf{C}) is observable if and only if for every monic polynomial $c(z)$ over $\mathcal{R}[z]$ of degree n there exists a matrix \mathbf{K} over \mathcal{R} of size $(n \times p)$, such that the matrix $[\mathbf{A} + \mathbf{K}\mathbf{C}]$ has characteristic polynomial $c(z)$ [20].

The standard decomposition of unobservable systems given in the following Lemma was first published by Kalman in [24] and can be also found in any standard text of linear systems theory.

Lemma 3: Let \mathbf{A} and \mathbf{C} be matrices over \mathcal{R} of size $(n \times n)$ and $(p \times n)$, respectively and \mathbf{C} not zero. Further, let the pair (\mathbf{A}, \mathbf{C}) is unobservable. Then there exists a non-singular matrix \mathbf{T} of size $(n \times n)$ such that

$$\begin{aligned} \mathbf{T}^{-1}\mathbf{A}\mathbf{T} &= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \\ \mathbf{C}\mathbf{T} &= [\mathbf{C}_1, \mathbf{0}] \end{aligned}$$

The pair $(\mathbf{A}_{11}, \mathbf{C}_1)$ is observable and the eigenvalues of the matrix \mathbf{A}_{22} are the unobservable eigenvalues of the pair (\mathbf{A}, \mathbf{C}) .

Lemma 4: Let \mathbf{A} and \mathbf{C} be matrices over \mathcal{R} of size $(n \times n)$ and $(p \times n)$, respectively and \mathbf{C} not zero. Further let

$$\mathbf{A} = \mathbf{T} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \mathbf{T}^{-1}, \quad \mathbf{C} = [\mathbf{C}_1, \mathbf{0}] \mathbf{T}^{-1}$$

with $(\mathbf{A}_{11}, \mathbf{C}_1)$ observable. If the pair (\mathbf{A}, \mathbf{C}) is detectable then the matrix \mathbf{A}_{22} is Schur stable [23].

Lemma 5: Let \mathbf{A} be a matrix over \mathcal{R} of size $(n \times n)$. Then the matrix \mathbf{A} is Schur stable if and only if the following condition holds:

(a) $\text{rank}[\mathbf{I}_n z - \mathbf{A}] = n, \forall z \in C^+$

Proof: Let \mathbf{A} be a strictly Schur matrix over \mathcal{R} . From Definition 2 it follows that the characteristic polynomial $c(z)$ of the matrix \mathbf{A} is a strictly Schur polynomial and therefore from Definition 1 it follows that

$$c(z) \neq 0, \forall z \in C^+ \tag{13}$$

The Smith-McMillan form of polynomial matrix $[\mathbf{I}_n z - \mathbf{A}]$ over $\mathcal{R}[z]$ is given by

$$\mathbf{K}(z) [\mathbf{I}_n z - \mathbf{A}] \mathbf{L}(z) = [\text{diag}[c_1(z), c_2(z), \dots, c_n(z)]] \quad (14)$$

where $\mathbf{K}(z)$ and $\mathbf{L}(z)$ are unimodular matrices over $\mathcal{R}[z]$. The polynomials $c_i(z)$ for $i=1, 2, \dots, n$ are the invariant polynomials of the matrix \mathbf{A} and therefore their product is the characteristic polynomial $c(z)$ of the matrix \mathbf{A} [20], that is

$$c(z) = \prod_{i=1}^n c_i(z) \quad (15)$$

From (13) and (15) it follows that

$$c_i(z) \neq 0, \forall z \in C^+, \forall i=1, 2, \dots, n \quad (16)$$

from (16) it follows that

$$\text{rank}\{\text{diag}[c_1(z), c_2(z), \dots, c_n(z)]\} = n, \forall z \in C^+ \quad (17)$$

Since $\mathbf{K}(z)$ and $\mathbf{L}(z)$ are unimodular matrices over $\mathcal{R}[z]$, from (12) and (14) we obtain:

$$\text{rank}[\mathbf{I}_n z - \mathbf{A}] = \text{rank}\{\text{diag}[c_1(z), c_2(z), \dots, c_n(z)]\} \quad (18)$$

Relationships (17) and (18) imply that

$$\text{rank}[\mathbf{I}_n z - \mathbf{A}] = n, \forall z \in C^+ \quad (19)$$

This is condition (a) of the Lemma. To prove sufficiency, we assume that condition (a) holds. Since by assumption condition (a) holds we have that

$$\text{rank}[\mathbf{I}_n z - \mathbf{A}] = n, \forall z \in C^+ \quad (20)$$

From (18) and (20) we obtain :

$$\text{rank}\{\text{diag}[c_1(z), c_2(z), \dots, c_n(z)]\} = n, \forall z \in C^+ \quad (21)$$

From (21) it follows that

$$c_i(z) \neq 0, \forall z \in C^+, \forall i=1, 2, \dots, n \quad (22)$$

From (22) it follows that

$$\prod_{i=1}^n c_i(z) \neq 0, \forall z \in C^+, \forall i=1, 2, \dots, n \quad (23)$$

Relationships (23) and (15) imply

$$c(z) \neq 0, \forall z \in C^+ \quad (24)$$

Relationship (24) and Definition 1 imply that $c(z)$ is a strictly Schur polynomial. Since by assumption $c(z)$ is the characteristic polynomial of the real matrix \mathbf{A} , from Definition 2 it follows that the matrix \mathbf{A} is Schur stable. This completes the proof.

The following Lemma is based on the results of [20].

Lemma 6: Let \mathbf{A} and \mathbf{C} be matrices over \mathcal{R} of size $(n \times n)$ and $(p \times n)$, respectively and \mathbf{C} not zero. Further, let the pair (\mathbf{A}, \mathbf{C}) be detectable. Then there exists a matrix \mathbf{K} over \mathcal{R} of size $(n \times p)$ such that the matrix $[\mathbf{A} + \mathbf{K}\mathbf{C}]$ is Schur stable.

Proof: Let the pair (\mathbf{A}, \mathbf{C}) be detectable. Detectability of the pair (\mathbf{A}, \mathbf{C}) implies that the pair (\mathbf{A}, \mathbf{C}) is either observable or unobservable with stable unobservable eigenvalues. If the pair (\mathbf{A}, \mathbf{C}) is observable, then from Lemma 2 it follows that there exists a matrix \mathbf{K} over \mathcal{R} of appropriate dimensions such that

$$\det[\mathbf{I}_n z - \mathbf{A} - \mathbf{K}\mathbf{C}] = \det[\mathbf{I}_n z - \mathbf{A} - \mathbf{K}\mathbf{C}] = c(z) \quad (25)$$

where $c(z)$ be an arbitrary monic, strictly Schur polynomial over $\mathcal{R}[z]$ of degree n . Since the notion of observability is a dual of reachability (i.e. observability of the pair (\mathbf{A}, \mathbf{C}) implies reachability of the pair $(\mathbf{A}^T, \mathbf{C}^T)$) [20], the matrix \mathbf{K} can be calculated using known methods for the solution of pole assignment problem by state feedback [20]. Since $c(z)$ is the characteristic polynomial of the matrix $[\mathbf{A} + \mathbf{K}\mathbf{C}]$, from Definition 2 and (25) it follows that the matrix $[\mathbf{A} + \mathbf{K}\mathbf{C}]$ is Schur stable. If the pair (\mathbf{A}, \mathbf{C}) is unobservable with stable unobservable eigenvalues, then from Lemma 3 and Lemma 4 it follows that there exists a non-singular matrix \mathbf{T} such that

$$\mathbf{A} = \mathbf{T} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \mathbf{T}^{-1}, \quad \mathbf{C} = [\mathbf{C}_1, \mathbf{0}] \mathbf{T}^{-1} \quad (26)$$

The pair $(\mathbf{A}_{11}, \mathbf{C}_1)$ is observable and the matrix \mathbf{A}_{22} is Schur stable. Schur stability of the matrix \mathbf{A}_{22} and Definition 2 imply that the polynomial $\chi(z)$ given by

$$\det[\mathbf{I}z - \mathbf{A}_{22}] = \chi(z) \quad (27)$$

is a strictly Schur polynomial. Observability of the pair $(\mathbf{A}_{11}, \mathbf{C}_1)$ and Lemma 2 imply the existence of a matrix \mathbf{K}_1 over \mathcal{R} of appropriate dimensions such that

$$\det[\mathbf{I}z - \mathbf{A}_{11} - \mathbf{K}_1 \mathbf{C}_1] = \varphi(z) \quad (28)$$

where $\varphi(z)$ is an arbitrary monic, strictly Schur polynomial over $\mathcal{R}[z]$ of appropriate degree. Since the notion of observability is a dual of reachability (i.e. observability of the pair $(\mathbf{A}_{11}, \mathbf{C}_1)$ implies reachability of the pair $(\mathbf{A}_{11}^T, \mathbf{C}_1^T)$), the matrix \mathbf{K}_1 can be calculated using known methods for the solution of pole assignment problem by state feedback [20]. Let

$$\mathbf{K} = \mathbf{T} \begin{bmatrix} \mathbf{K}_1 \\ \mathbf{0} \end{bmatrix} \quad (29)$$

Using (26) and (29) we have that

$$[\mathbf{A} + \mathbf{K}\mathbf{C}] = \mathbf{T} \begin{bmatrix} \mathbf{A}_{11} + \mathbf{K}_1\mathbf{C}_1 & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \mathbf{T}^{-1} \quad (30)$$

while from (27), (28) and (30) we have that

$$\det[(\mathbf{I}_n z - \mathbf{A} - \mathbf{K}\mathbf{C})] = \varphi(z)\chi(z) \quad (31)$$

Since by (27) and (28) the polynomials $\chi(z)$ and $\varphi(z)$ are strictly Schur, the polynomial $[\varphi(z)\chi(z)]$ is also a strictly Schur polynomial. Since by (31) the polynomial $[\varphi(z)\chi(z)]$ is the characteristic polynomial of the matrix $[\mathbf{A} + \mathbf{K}\mathbf{C}]$, from Definition 2 it follows that the matrix $[\mathbf{A} + \mathbf{K}\mathbf{C}]$ is Schur stable. This completes the proof.

4. MAIN RESULTS

The theorem that follows is the main result of this paper and gives the necessary and sufficient conditions for the existence of a full order proportional-integral observer for the state estimation of discrete-time linear time-invariant systems.

Theorem 1. The system described by equations (4) and (5) is a proportional-integral observer of order n of the system described by equations (1) and (2), if and only if the following condition holds:

(a) The pair (\mathbf{A}, \mathbf{C}) is detectable.

Proof: Let the system described by equations (4) and (5) is a proportional-integral observer of order n of the system described by equations (1) and (2). Then the real matrix of size $((n+p) \times (n+p))$ given by (8) is Schur stable. Schur stability of the matrix given by (8) and Lemma 5 imply

$$\text{rank} \begin{bmatrix} \mathbf{I}_n z - \mathbf{A} + \mathbf{L}\mathbf{C} & -\mathbf{F} \\ \mathbf{C} & (z-1)\mathbf{I}_p \end{bmatrix} = (n+p), \forall z \in C^+ \quad (32)$$

Since the $(n+p)$ columns of the matrix on the left side of (32) are linearly independent over C , $\forall z \in C^+$, a subset of these columns consisting of the first n columns must be also linearly independent over C , $\forall z \in C^+$; therefore

$$\text{rank} \begin{bmatrix} \mathbf{I}_n z - \mathbf{A} + \mathbf{L}\mathbf{C} \\ \mathbf{C} \end{bmatrix} = \text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n z - \mathbf{A} + \mathbf{L}\mathbf{C} \end{bmatrix} = n, \forall z \in C^+ \quad (33)$$

From (33) and after simple algebraic manipulations we obtain:

$$\text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n z - \mathbf{A} + \mathbf{L}\mathbf{C} \end{bmatrix} = \text{rank} \left\{ \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{L} & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n z - \mathbf{A} \end{bmatrix} \right\} = n, \forall z \in C^+ \quad (34)$$

Since matrix

$$\begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{L} & \mathbf{I}_n \end{bmatrix} \quad (35)$$

is non-singular, from (34) it follows that

$$\text{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n z - \mathbf{A} \end{bmatrix} = n, \forall z \in \mathbb{C}^+ \quad (36)$$

Relationship (36) and condition (a) of Lemma 1 imply that the pair (\mathbf{A}, \mathbf{C}) is detectable. This is condition (a) of the Theorem.

To prove sufficiency, we assume that condition (a) holds. Detectability of the pair (\mathbf{A}, \mathbf{C}) and Lemma 6 imply the existence a matrix \mathbf{K} over \mathcal{R} of size $(n \times p)$ such that the matrix $[\mathbf{A} + \mathbf{K}\mathbf{C}]$ is Schur stable, that is

$$\det[\mathbf{I}_n z - \mathbf{A} - \mathbf{K}\mathbf{C}] = c(z) \quad (37)$$

where $c(z)$ is a strictly Schur polynomial over $\mathcal{R}[z]$ of degree n . Matrix \mathbf{K} in (37) can be calculated as in the proof of Lemma 6.

From (3) it follows that there exists a non-singular matrix \mathbf{T} of size $(n \times n)$ such that

$$\mathbf{C} = [\mathbf{I}_p, \mathbf{0}]\mathbf{T} \quad (38)$$

Let Φ be an arbitrary nonzero Schur stable matrix over \mathcal{R} of size $(p \times p)$. Furthermore, let \mathbf{X} be a matrix over \mathcal{R} of size $(n \times p)$ given by

$$\mathbf{X} = \mathbf{T}^{-1} \begin{bmatrix} (-\Phi + \mathbf{I}_p) \\ \Lambda \end{bmatrix} \quad (39)$$

where Λ is an arbitrary matrix over \mathcal{R} of size $((n-p) \times p)$. From (38) and (39) we have:

$$-\mathbf{C}\mathbf{X} + \mathbf{I}_p = [-\mathbf{I}_p, \mathbf{0}]\mathbf{T}\mathbf{T}^{-1} \begin{bmatrix} (-\Phi + \mathbf{I}_p) \\ \Lambda \end{bmatrix} + \mathbf{I}_p = \Phi \quad (40)$$

Now we form the matrix \mathbf{M} over \mathcal{R} of size $((n+p) \times (n+p))$ [25]

$$\mathbf{M} = \begin{bmatrix} \mathbf{I}_n & \mathbf{X} \\ \mathbf{0} & \mathbf{I}_p \end{bmatrix}$$

Matrix \mathbf{M} over \mathcal{R} is non-singular and its inverse is given by

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{I}_n & -\mathbf{X} \\ \mathbf{0} & \mathbf{I}_p \end{bmatrix}$$

Then, we obtain:

$$\mathbf{M}^{-1} \begin{bmatrix} \mathbf{A} - \mathbf{L}\mathbf{C} & \mathbf{F} \\ -\mathbf{C} & \mathbf{I}_p \end{bmatrix} \mathbf{M} = \begin{bmatrix} \mathbf{A} + (\mathbf{X} - \mathbf{L})\mathbf{C} & (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{X} + \mathbf{F} - \mathbf{X}(-\mathbf{C}\mathbf{X} + \mathbf{I}_p) \\ -\mathbf{C} & -\mathbf{C}\mathbf{X} + \mathbf{I}_p \end{bmatrix} \quad (41)$$

Furthermore, we set:

$$\mathbf{L} = \mathbf{x} - \mathbf{K} \quad (42)$$

$$\mathbf{F} = -(\mathbf{A} - \mathbf{LC})\mathbf{X} + \mathbf{X}(-\mathbf{CX} + \mathbf{I}_p) \quad (43)$$

Now by substituting (40), (42) and (43) into (41) we have :

$$\mathbf{M}^{-1} \begin{bmatrix} \mathbf{A} - \mathbf{LC} & \mathbf{F} \\ -\mathbf{C} & \mathbf{I}_p \end{bmatrix} \mathbf{M} = \begin{bmatrix} \mathbf{A} + \mathbf{KC} & \mathbf{0} \\ -\mathbf{C} & \mathbf{\Phi} \end{bmatrix} \quad (44)$$

Since by (37) matrix $[\mathbf{A} + \mathbf{KC}]$ is Schur stable and the nonzero real matrix $\mathbf{\Phi}$ is by assumption Schur stable, from Lemma 5 it follows that

$$\text{rank}[\mathbf{I}_n z - \mathbf{A} - \mathbf{KC}] = n, \forall z \in \mathcal{C}^+ \quad (45)$$

$$\text{rank}[\mathbf{I}_p z - \mathbf{\Phi}] = p, \forall z \in \mathcal{C}^+ \quad (46)$$

Relationships (45) and (46) imply

$$\text{rank} \begin{bmatrix} \mathbf{I}_n z - \mathbf{A} - \mathbf{KC} & \mathbf{0} \\ \mathbf{C} & \mathbf{I}_p z - \mathbf{\Phi} \end{bmatrix} = (n + p), \forall z \in \mathcal{C}^+ \quad (47)$$

Since by (44) matrices

$$\begin{bmatrix} \mathbf{A} - \mathbf{LC} & \mathbf{F} \\ -\mathbf{C} & \mathbf{I}_p \end{bmatrix}, \begin{bmatrix} \mathbf{A} + \mathbf{KC} & \mathbf{0} \\ -\mathbf{C} & \mathbf{\Phi} \end{bmatrix} \quad (48)$$

of size $((n+p) \times (n+p))$ are similar, we have:

$$\text{rank} \begin{bmatrix} \mathbf{I}_n z - \mathbf{A} + \mathbf{LC} & -\mathbf{F} \\ \mathbf{C} & (z-1)\mathbf{I}_p \end{bmatrix} = \text{rank} \begin{bmatrix} \mathbf{I}_n z - \mathbf{A} - \mathbf{KC} & \mathbf{0} \\ \mathbf{C} & \mathbf{I}_p z - \mathbf{\Phi} \end{bmatrix} \quad (49)$$

From (47) and (49), we obtain:

$$\text{rank} \begin{bmatrix} \mathbf{I}_n z - \mathbf{A} + \mathbf{LC} & -\mathbf{F} \\ \mathbf{C} & (z-1)\mathbf{I}_p \end{bmatrix} = (n + p), \forall z \in \mathcal{C}^+ \quad (50)$$

Furthermore, from (50) and Lemma 5 it follows that matrix

$$\begin{bmatrix} \mathbf{A} - \mathbf{LC} & \mathbf{F} \\ -\mathbf{C} & \mathbf{I}_p \end{bmatrix}$$

with \mathbf{L} and \mathbf{F} given by (42), (43) respectively is Schur stable and therefore according to (8) the system described by equations (4) and (5) with \mathbf{L} and \mathbf{F} given by (42) and (43), respectively is a proportional-integral observer of order n of the system described by equations (1) and (2). This completes the proof.

The sufficiency part of the proof of Theorem 1 provides a construction of the matrices \mathbf{L} and \mathbf{F} of proportional-integral observer of order n for the system described by equations (1) and (2). The major steps of this construction are given below.

CONSTRUCTION

Given : \mathbf{A} , \mathbf{B} and \mathbf{C}

Find : \mathbf{L} and \mathbf{F}

Step 1: Check condition (a) of Theorem 1. If this condition is satisfied go to *Step 2*. If condition (a) of Theorem 1 is not satisfied, the construction of a proportional-integral observer of order n is impossible.

Step 2: Delectability of the pair (\mathbf{A}, \mathbf{C}) and Lemma 6 imply the existence of a matrix \mathbf{K} over \mathcal{R} of size $(n \times p)$ such that the matrix $[\mathbf{A} + \mathbf{K}\mathbf{C}]$ is Hurwitz stable. The matrix \mathbf{K} can be calculated as in the proof of Lemma 6.

Step 3: Find a non-singular matrix \mathbf{T} of size $(n \times n)$ such that

$$\mathbf{C} = [\mathbf{I}_p, \mathbf{0}]\mathbf{T}$$

Step 4: Let Φ be an arbitrary nonzero Schur stable matrix over \mathcal{R} of size $(p \times p)$. Further, let Λ be an arbitrary matrix over \mathcal{R} of size $((n-p) \times p)$, where we set:

$$\mathbf{X} = \mathbf{T}^{-1} \begin{bmatrix} (-\Phi + \mathbf{I}_p) \\ \Lambda \end{bmatrix}$$

$$\mathbf{L} = \mathbf{X} - \mathbf{K}$$

$$\mathbf{F} = -(\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{X} + \mathbf{X}(-\mathbf{C}\mathbf{X} + \mathbf{I}_p)$$

5. CONCLUSIONS

In this paper, by using basic concepts and basic results from linear systems and control theory as well as of the theory of matrices, the problem of the design of a full order proportional-integral observer for the state estimation of discrete-time linear time-invariant systems is studied and completely solved. The proof of the main results of this paper is constructive and furnishes a simple procedure for the construction of full order proportional-integral observer.

REFERENCES

- [1] D. G. Luenberger, "An Introduction to Observers", IEEE Trans. Autom. Control vol. 16, no. 5, pp. 596-602, 1971
- [2] B. Wojciechowski, "Analysis and Synthesis of Proportional-Integral Observers for single-input single output time-invariant continuous systems", Ph.D. dissertation, Technical University of Gliwice, Poland, 1978.
- [3] T. Kaczorek, "Proportional-integral observers for linear multivariable time-varying Systems", Regelungstechnik, vol. 27, no. 1, pp. 359-362, 1979.
- [4] B. Shafai and R.L. Carroll, "Design of proportional integral observer for linear time varying multivariable systems", Proceedings of 24th IEEE conference on Decision and control, pp. 597-599, 1985.
- [5] S. Beale and B. Shafai, "Robust control system design with a proportional integral observer", International Journal of Control, vol. 50, no. 1, pp. 97-111, 1989.
- [6] H.H. Niemann, J. Stoustrup, B. Shafai and S. Beale, "LTR design of proportional integral observers", International Journal Robust Nonlinear Control, vol. 5, no.7, pp. 671-693, 1995.
- [7] B. Shafai, S. Beale, H. H. Niemann and J. Stoustrup, "LTR design of discrete-time proportional-integral observers", IEEE Transactions Automatic Control, vol. 41, no. 7, pp. 1056-1062, 1996.
- [8] D. Söffker, T. J. Yu and P. C. Muller, "State estimation of dynamical systems with non-linearities by using proportional-integral observer", International Journal of Systems Science, vol. 26, no. 9, pp. 1571-1582, 1995.
- [9] K. K. Busawon and P. Kabore, "On the design of integral and proportional integral observers", Proceedings of the American control conference, pp.3725-3729, 2000.
- [10] G. R. Duan, G. P. Liu and S. Thompson, "Eigenstructure assignment design for proportional integral observers: continuous-time case", IEE Proceedings. Contr. Theory Applications, vol. 148, no. 3, pp. 263-267, 2001.
- [11] G. R. Duan, G. P. Liu and S. Thompson, "Eigenstructure assignment design for proportional-integral observers: the discrete-time case", International Journal of Systems Science, vol. 34, no. 5, pp. 357-363, 2003.
- [12] F. Bakhshande, R. Bach and D. Söffker, "Robust control of a hydraulic cylinder using an observer-based sliding mode control: Theoretical development and experimental validation, Control Engineering Practice, vol. 95, 104272, 2020.
- [13] T. Białon, M. Pasko, and R. Niestroj, "Developing induction motor state observers with increased robustness", *Energies*, vol. 13, no. 20, pp. 54-87, 2020.
- [14] B. Shafai, A. Moradmand, "Design of an Integrated Observer Structure for robust Fault Detection, IEEE Conference on Control Technology and Applications, pp. 24-26, 2020.
- [15] Y. Hu, X. Dai, D. Cui and Z. Jia, "Zero-assignment and complex coefficient gain design of generalized proportional-integral observer for robust fault detection", Journal Franklin Institute, vol. 359, no. 8, pp. 3831-3856, 2022.
- [16] L. Lin, "Fault detection in uncertain systems via proportional integral observer" Systems Science & Control Engineering, vol. 10, no. 1, pp. 822-831, 2022.
- [17] F. Bakhshande and D. Söffker, "Proportional integral-observer: A brief survey with special attention to the actual methods using acc benchmark", IFAC papers online, vol. 48, no. 1, pp. 532-537, 2015.

- [18] Y. Liu, Robust Nonlinear Control Design with Proportional-Integral Observer Technique, Ph.D. dissertation, Duisburg University, Germany, 2011.
- [19] F. Bakhshande, Observer-Based Robust Nonlinear Control Design, Ph.D. dissertation, Duisburg University, Germany, 2018.
- [20] V. Kucera, Analysis and Design of Discrete Linear Control Systems, Prentice Hall, London, 1991.
- [21] W. M. Wonham, “On a matrix Riccati equation of stochastic control”, SIAM J. Control and Optimization, vol. 6, no. 4, pp. 681-697, 1968.
- [22] H. L. Trentelman, A. A. Stoorvogel and M. Hautus, Control Theory for Linear Systems, Springer-Verlag, London, 2002.
- [23] K. Zhou, J. C. Doyle and K. Glover, Optimal and Robust Control, Prentice-Hall, New Jersey, 1996.
- [24] R. E. Kalman, “Mathematical description of linear dynamical systems”, SIAM J. Control and Optimization, vol. 1, no. 2, pp. 152-192, 1963.
- [25] K. Takahashi, “Eigenvalues of matrices with given block upper triangular part”, Linear Algebra and its Applications, vol. 239, pp. 175-184, 1996.