STUDY OF FIELD SOURCE STRUCTURES DESCRIBED BY CIRCULAR CYLINDER WAVE FUNCTIONS

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Abstract

The structures of field sources described by the wave functions of a circular cylinder have been investigated. These sources exhibit the character of two-dimensional multipoles, consisting of a set of monopoles confined within a small planar region. Multiple possible configurations have been identified for each multipole source, differing in geometric arrangement, the number of monopoles and the accuracy with which they represent the original wave field.

1. INTRODUCTION

The work focuses on investigating the structures of sources of fields which are described by the wave functions of a circular cylinder [1]

$$H_n^{(1)}(k\rho)\cos(n\varphi),\ H_n^{(1)}(k\rho)\sin(n\varphi). \tag{1}$$

Here k is a wave number, (ρ, φ) are the polar coordinates of the observation point, $H_n^{(1)}(k\rho)$ is a Hankel function and n is a non-negative integer. As it is known, the functions (1) are the solutions of the two dimensional Helmholtz equation. They are used for studying the problems, related to the harmonic in time wave processes, when the area is bounded by the circular cylinder surface. It is supposed, that the time factor is $e^{-i\omega t}$.

The corresponding zero-order function $H_0^{(1)}(k\rho)$ describes the field of the elementary two dimensional source (monopole), which is propagating as a travelling cylindrical wave, from the origin. Since the expression of this field depends only on the coordinate ρ, then its amplitude radiation pattern will have the shape of a circle. The situation changes for the fields (1) at subsequent values of n. Since they also depend on the angular coordinate φ, then their radiation patterns have more complicated shape (Figure 1), which indicates the multipole nature of the sources of such fields. The objective is to analyze the structures of these sources.

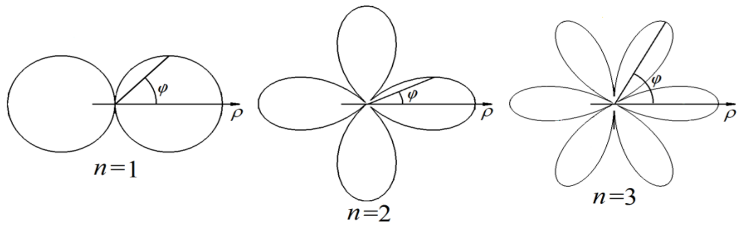
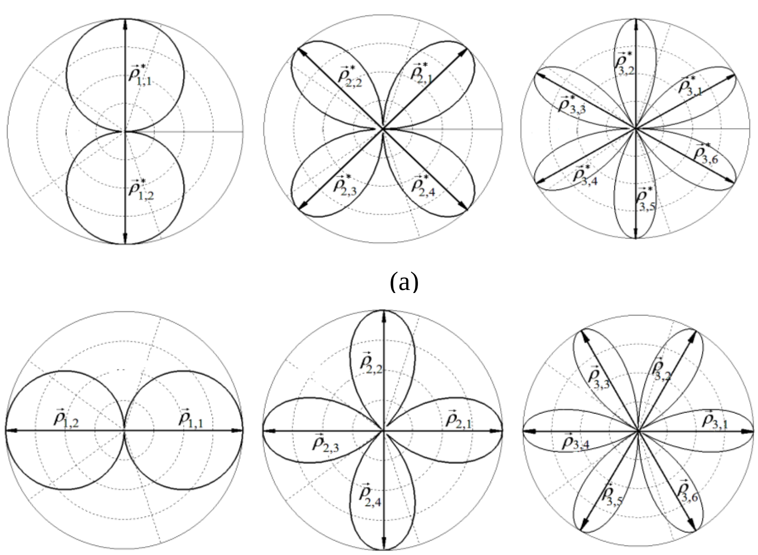


Figure 1. Examples of radiation patterns for $H_n^{(1)}(k\rho)\cos(n\varphi)$

This paper proposes two main approaches to solving the problem. The first one is based on the consideration the amplitude radiation patterns of fields (1) and the construction the linear differential operators. Their effect on the $H_0^{(1)}(k\rho)$ yields to the original fields [2]. The second one is based on the application of the known addition theorem for the Bessel cylindrical functions. As a result of investigation, four types of multipoles were identified. They differ by the geometric construction, number of monopoles and also by the accuracy of the original field’s representation.

2. PROBLEM STATEMENT

Let us focus on the radiation patterns of the fields (1), at initial values of n =1,2,3,…(Figure 2).



(a)

(b)

Figure 2. Examplesof radiation patterns: (a) $H_n^{(1)}(k\rho)\cos(n\varphi)$ and (b) $H_n^{(1)}(k\rho)\sin(n\varphi)$

It can be seen, that each of them consist of 2n lobes, the directions of which can be determined by the unit vectors

$$\vec{\rho}_{n,j}=\left\{\cos\varphi_{n,j},\sin\varphi_{n,j}\right\},\ \vec{\rho}^{\,*}_{n,j}=\left\{\cos\varphi^*_{n,j},\sin\varphi^*_{n,j}\right\},\tag{2}$$

Where

$$\varphi_{n,j}=\frac{\pi}{n}(j-1),\ \varphi^*_{n,j}=\frac{\pi}{n}\left(j-\frac{1}{2}\right),\ j=1,...,2n.\tag{3}$$

2. 1. First Kind Operators

Let consider the linear differential operators with the form

$$\hat{L}_n=\frac{(-1)^n 2^{n-1}}{nk^n}\sum_{j=1}^{n}(-1)^{j+1}\frac{\partial^n}{\partial\vec{\rho}_{n,j}^{\,n}},\ \hat{L}^*_n=\frac{(-1)^n 2^{n-1}}{nk^n}\sum_{j=1}^{n}(-1)^{j+1}\frac{\partial^n}{\partial\vec{\rho}^{\,*\,n}_{n,j}}.\tag{4}$$

Here, under the summation sign, there are derivative operators of n-th order along the directions of the vectors (2), i.e.

$$\frac{\partial^n}{\partial\vec{\rho}_{n,j}^{\,n}}=\left(\cos\varphi_{n,j}\frac{\partial}{\partial x}+\sin\varphi_{n,j}\frac{\partial}{\partial y}\right)^n,\ \frac{\partial^n}{\partial\vec{\rho}^{\,*\,n}_{n,j}}=\left(\cos\varphi^*_{n,j}\frac{\partial}{\partial x}+\sin\varphi^*_{n,j}\frac{\partial}{\partial y}\right)^n.$$

Applying Newton's binomial formula and introducing the notation

$$B_{n,q}=\sum_{j=1}^{n}(-1)^{j+1}\left(\cos\varphi_{n,j}\right)^{n-q}\left(\sin\varphi_{n,j}\right)^q,\ B^*_{n,q}=\sum_{j=1}^{n}(-1)^{j+1}\left(\cos\varphi^*_{n,j}\right)^{n-q}\left(\sin\varphi^*_{n,j}\right)^q,$$

operators (4) can be represented as

$$\hat{L}_n=\frac{(-1)^n 2^{n-1}}{nk^n}\sum_{q=0}^{n}C_n^q B_{n,q}\frac{\partial^n}{\partial x^{n-q}\partial y^q},\ \hat{L}^*_n=\frac{(-1)^n 2^{n-1}}{nk^n}\sum_{q=0}^{n}C_n^q B^*_{n,q}\frac{\partial^n}{\partial x^{n-q}\partial y^q}.$$

Calculating the $B_{n,q}$ and $B^*_{n,q}$ coefficients, for the different values of n, it can be noted that they can be determined as

$$B_{n,q}=(-1)^{\lfloor q/2\rfloor}\left[1+(-1)^q\right]\left(n/2^n\right),\ B^*_{n,q}=(-1)^{\lfloor q/2\rfloor}\left[1-(-1)^q\right]\left(n/2^n\right),$$

where $\lfloor q/2\rfloor$ is the integer part of the number $q/2$. Then, taking into account that in the expressions of the operators $\hat{L}_n$ and $\hat{L}^*_n$ the terms only with even and odd values of q respectively must remain, then finally, we obtain

$$\hat{L}_n=\frac{(-1)^n}{k^n}\sum_{\alpha=0}^{\lfloor n/2\rfloor}(-1)^\alpha C_n^{2\alpha}\frac{\partial^n}{\partial x^{n-2\alpha}\partial y^{2\alpha}},\ \hat{L}^*_n=\frac{(-1)^n}{k^n}\sum_{\beta=1}^{\lfloor (n+1)/2\rfloor}(-1)^{\beta+1}C_n^{2\beta-1}\frac{\partial^n}{\partial x^{n-2\beta+1}\partial y^{2\beta-1}}.\tag{5}$$

Let us now consider a homogeneous polynomial of degree n, from variables x, y,

$$f_n(x,y) = \sum_{\alpha=0}^{n} A_{n,\alpha} x^{n-\alpha} y^{\alpha} ,$$

where $A_{n,\alpha}$ are the known coefficients. We will say that the linear differential operator $\hat{F}_n$ is the eigen-operator for this polynomial, if it is the result of the following transformation:

$$x \to -\frac{1}{k}\frac{\partial}{\partial x}, \; y \to -\frac{1}{k}\frac{\partial}{\partial y} .$$

So, it will have the form

$$\hat{F}_n = \frac{(-1)^n}{k^n} \sum_{\alpha=0}^{n} A_{n,\alpha} \frac{\partial^n}{\partial x^{n-\alpha} \partial y^{\alpha}} . \tag{6}$$

Let us show, that $\hat{L}_n$ and $\hat{L}_n^*$ are the eigen-operators for the cylindrical harmonics $\rho^n \cos(n\varphi)$ and $\rho^n \sin(n\varphi)$ [3]. For this, we will use well-known expression for the power of a complex number:

$$\rho^n e^{\pm in\varphi} = (x \pm iy)^n = \sum_{\alpha=0}^{\lfloor n/2 \rfloor} (-1)^{\alpha} C_n^{2\alpha} x^{n-2\alpha} y^{2\alpha} \pm i \sum_{\beta=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{\beta+1} C_n^{2\beta-1} x^{n-2\beta+1} y^{2\beta-1} ,$$

from where follows the next homogeneous polynomials

$$\rho^n \cos(n\varphi) = \sum_{\alpha=0}^{\lfloor n/2 \rfloor} (-1)^{\alpha} C_n^{2\alpha} x^{n-2\alpha} y^{2\alpha} , \; \rho^n \sin(n\varphi) = \sum_{\beta=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{\beta+1} C_n^{2\beta-1} x^{n-2\beta+1} y^{2\beta-1} .$$

The eigen-operators of these polynomials, according to the definition (6) will coincide with $\hat{L}_n$ and $\hat{L}_n^*$. For the eigen-operator of polynomial $(x \pm iy)^n$ we have the expression

$$\frac{(-1)^n}{k^n} \left(\frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y} \right)^n = \hat{L}_n \pm i \hat{L}_n^* . \tag{7}$$

The operators $\hat{L}_n$ and $\hat{L}_n^*$ give ability to connect the original *n*-order fields (1) with the zero-order field $H_0^{(1)}(k\rho)$. Namely, we will show that for any $n \geq 1$,

$$\hat{L}_n H_0^{(1)}(k\rho) = H_n^{(1)}(k\rho)\cos(n\varphi), \; \hat{L}_n^* H_0^{(1)}(k\rho) = H_n^{(1)}(k\rho)\sin(n\varphi) . \tag{8}$$

For this, we combine functions (1) as

$$S_n^{\pm}(\rho,\varphi) = H_n^{(1)}(w) e^{\pm in\varphi} , \tag{9}$$

where $w = k\rho$ and consider the partial derivatives of function $S_n^{\pm}(\rho,\varphi)$ by *x* and *y*. After some transformations we will have

$$\frac{1}{k}\frac{\partial S_n^{\pm}(\rho,\varphi)}{\partial x} = \left[\frac{dH_n^{(1)}(w)}{dw}\cos\varphi \mp i\frac{n}{w}H_n^{(1)}(w)\sin\varphi\right]e^{\pm in\varphi},$$

$$\frac{1}{k}\frac{\partial S_n^{\pm}(\rho,\varphi)}{\partial y} = \left[\frac{dH_n^{(1)}(w)}{dw}\sin\varphi \pm i\frac{n}{w}H_n^{(1)}(w)\cos\varphi\right]e^{\pm in\varphi},$$

from where we can write

$$\frac{1}{k}\frac{\partial S_n^{\pm}(\rho,\varphi)}{\partial x} \pm i\frac{1}{k}\frac{\partial S_n^{\pm}(\rho,\varphi)}{\partial y} = \left[\frac{dH_n^{(1)}(w)}{dw} - \frac{n}{w}H_n^{(1)}(w)\right]e^{\pm i(n+1)\varphi},$$

or taking into account known recurrent formula [4]

$$\frac{dH_n^{(1)}(w)}{dw} - \frac{n}{w}H_n^{(1)}(w) = -H_{n+1}^{(1)}(w), \tag{10}$$

we finally obtain:

$$-\frac{1}{k}\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right)S_n^{\pm}(\rho,\varphi) = S_{n+1}^{\pm}(\rho,\varphi). \tag{11}$$

As a special case, when *n* =0, we have

$$-\frac{1}{k}\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right)S_0^{\pm}(\rho,\varphi) = S_1^{\pm}(\rho,\varphi).$$

Similarly, when *n* =1,

$$-\frac{1}{k}\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right)S_1^{\pm}(\rho,\varphi) = S_2^{\pm}(\rho,\varphi),$$

or

$$\frac{(-1)^2}{k^2}\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right)^2 S_0^{\pm}(\rho,\varphi) = S_2^{\pm}(\rho,\varphi).$$

If we continue this process, in general we can write

$$\frac{(-1)^n}{k^n}\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right)^n S_0^{\pm}(\rho,\varphi) = S_n^{\pm}(\rho,\varphi),$$

or considering that $S_0^{\pm}(\rho,\varphi) = H_0^{(1)}(w)$, using (7) and (9),

$$\left(\hat{L}_n \pm i\hat{L}_n^*\right)H_0^{(1)}(k\rho) = H_n^{(1)}(k\rho)e^{\pm in\varphi}.$$

The last expression is equivalent to two expressions

$$\hat{L}_n H_0^{(1)}(k\rho)+i\hat{L}_n^* H_0^{(1)}(k\rho)=H_n^{(1)}(k\rho)\cos(n\varphi)+iH_n^{(1)}(k\rho)\sin(n\varphi),$$

$$\hat{L}_n H_0^{(1)}(k\rho)-i\hat{L}_n^* H_0^{(1)}(k\rho)=H_n^{(1)}(k\rho)\cos(n\varphi)-iH_n^{(1)}(k\rho)\sin(n\varphi),$$

from which equalities (8) follow. In expanded form we can write

$$\frac{(-1)^n 2^{n-1}}{nk^n}\sum_{j=1}^{n}(-1)^{j+1}\frac{\partial^n H_0^{(1)}(k\rho)}{\partial\vec{\rho}_{n,j}^{\;n}}=H_n^{(1)}(k\rho)\cos(n\varphi), \tag{12}$$

$$\frac{(-1)^n 2^{n-1}}{nk^n}\sum_{j=1}^{n}(-1)^{j+1}\frac{\partial^n H_0^{(1)}(k\rho)}{\partial\vec{\rho}_{n,j}^{*\;n}}=H_n^{(1)}(k\rho)\sin(n\varphi). \tag{13}$$

2. 2. Second Kind Operators

Let us now consider operators of the form

$$\hat{M}_n=\frac{2^{n-2}(n-1)!}{(k\rho_0)^n}\sum_{j=1}^{2n}(-1)^{j+1}\sum_{m=0}^{n}\frac{(-1)^m\rho_0^{\;m}}{m!}\frac{\partial^m}{\partial\vec{\rho}_{n,j}^{\;m}}, \tag{14}$$

$$\hat{M}_n^*=\frac{2^{n-2}(n-1)!}{(k\rho_0)^n}\sum_{j=1}^{2n}(-1)^{j+1}\sum_{m=0}^{n}\frac{(-1)^m\rho_0^{\;m}}{m!}\frac{\partial^m}{\partial\vec{\rho}_{n,j}^{*\;m}}, \tag{15}$$

where ρ_0 is some coefficient. Applying Newton's binomial formula, after a series of transformations, these operators can be written as

$$\hat{M}_n=\frac{2^{n-1}(n-1)!}{(k\rho_0)^n}\sum_{m=0}^{n}\frac{\rho_0^{\;m}}{m!}\sum_{q=0}^{m}C_m^q B_{n,m,q}\frac{\partial^m}{\partial x^{m-q}\partial y^q},\;\hat{M}_n^*=\frac{2^{n-1}(n-1)!}{(k\rho_0)^n}\sum_{m=0}^{n}\frac{\rho_0^{\;m}}{m!}\sum_{q=0}^{m}C_m^q B_{n,m,q}^*\frac{\partial^m}{\partial x^{m-q}\partial y^q},$$

where the following notations are introduced

$$B_{n,m,q}=\frac{1}{2}\left[(-1)^m+(-1)^n\right]\sum_{j=1}^{n}(-1)^{j+1}\left(\cos\varphi_{n,j}\right)^{m-q}\left(\sin\varphi_{n,j}\right)^q,$$

$$B_{n,m,q}^*=\frac{1}{2}\left[(-1)^m+(-1)^n\right]\sum_{j=1}^{n}(-1)^{j+1}\left(\cos\varphi_{n,j}^*\right)^{m-q}\left(\sin\varphi_{n,j}^*\right)^q.$$

Calculating the values of the coefficients $B_{n,m,q}$ and $B_{n,m,q}^*$ for different *n*, we can notice that

$$B_{n,m,q}=(-1)^n\delta_{mn}B_{n,q},\;B_{n,m,q}^*=(-1)^n\delta_{mn}B_{n,q}^*,$$

where δ_{mn} is Kronecker delta. As a result, the operators $\hat{M}_n$ and $\hat{M}_n^*$ will be transformed to the form

$$\hat{M}_n = \frac{(-1)^n 2^{n-1}}{nk^n} \sum_{q=0}^{n} C_n^q B_{n,q} \frac{\partial^n}{\partial x^{n-q} \partial y^q}, \ \hat{M}_n^* = \frac{(-1)^n 2^{n-1}}{nk^n} \sum_{q=0}^{n} C_n^q B_{n,q}^* \frac{\partial^n}{\partial x^{n-q} \partial y^q},$$

i.e. they coincide with the first type operators $\hat{L}_n$ and $\hat{L}_n^*$. Thus, similarly to expressions (12) and (13), we write

$$\frac{2^{n-2}(n-1)!}{(k\rho_0)^n} \sum_{j=1}^{2n} (-1)^{j+1} \sum_{m=0}^{n} \frac{(-1)^m \rho_0{}^m}{m!} \frac{\partial^m H_0^{(1)}(k\rho)}{\partial \vec{\rho}_{n,j}{}^m} = H_n^{(1)}(k\rho)\cos(n\varphi), \tag{16}$$

$$\frac{2^{n-2}(n-1)!}{(k\rho_0)^n} \sum_{j=1}^{2n} (-1)^{j+1} \sum_{m=0}^{n} \frac{(-1)^m \rho_0{}^m}{m!} \frac{\partial^m H_0^{(1)}(k\rho)}{\partial \vec{\rho}^{\,*}_{n,j}{}^m} = H_n^{(1)}(k\rho)\sin(n\varphi). \tag{17}$$

3. FIRST TYPE CIRCULAR MULTIPOLE

Let us first consider expressions (12) and (13). Let us replace the directional derivatives in them by the corresponding central finite differences [5]. If the difference step is denoted by ρ_0, then for the outer area ($\rho > (n/2)\rho_0$) we will have the approximate equalities

$$\frac{2^{n-1}}{n(k\rho_0)^n} \sum_{j=1}^{n} \sum_{m=0}^{n} (-1)^{m+n+j+1} C_n^m H_0^{(1)}\left(k\left|\vec{\rho} - \left(m - \frac{n}{2}\right)\rho_0 \vec{\rho}_{n,j}\right|\right) \approx H_n^{(1)}(k\rho)\cos(n\varphi), \tag{18}$$

$$\frac{2^{n-1}}{n(k\rho_0)^n} \sum_{j=1}^{n} \sum_{m=0}^{n} (-1)^{m+n+j+1} C_n^m H_0^{(1)}\left(k\left|\vec{\rho} - \left(m - \frac{n}{2}\right)\rho_0 \vec{\rho}^{\,*}_{n,j}\right|\right) \approx H_n^{(1)}(k\rho)\sin(n\varphi), \tag{19}$$

where $\vec{\rho}$ is the radius vector of the observation point, with coordinates (ρ, φ).The left parts of these expressions describe the multipole field, the monopoles of which are located in points with radius vectors $(m - n/2)\rho_0 \vec{\rho}_{n,j}$ or $(m - n/2)\rho_0 \vec{\rho}^{\,*}_{n,j}$ respectively and have the amplitudes $(2^{n-1}/n)(k\rho_0)^{-n}(-1)^{m+n+j+1} C_n^m$. The radius of a given multipole is determined as $(n/2)\rho_0$, i.e. for the fixed value of ρ_0, increases with increasing n. The total number of the monopoles depends on the parity of n. For odd n, it is $n(n+1)$. For even n, some monopoles end up in the center, which corresponds to one monopole 1of total amplitude. The total number of monopoles, in this case, is n^2+1. The structure of the resulting multipole, which we will call the first type circular multipole, is shown on the Figure 3. As we see, the monopoles are located on concentric circles, along the lobes of the corresponding field pattern.

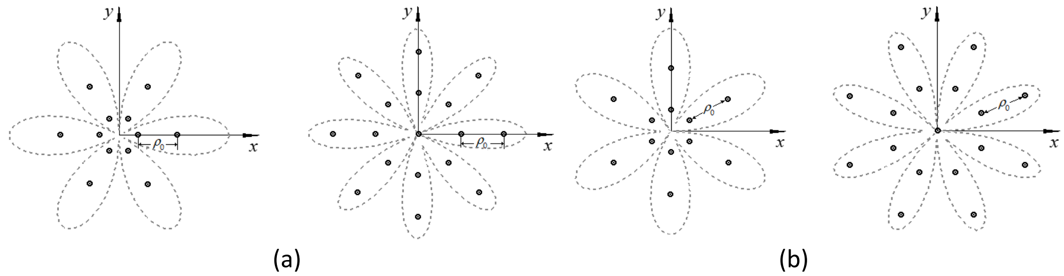


Figure 3. The first type of circular multipole, which describes the field: (a) $H_n^{(1)}(k\rho)\cos(n\varphi)$and (b) $H_n^{(1)}(k\rho)\sin(n\varphi)$, when n=3 and n=4

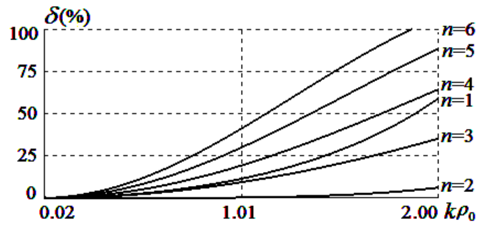


Figure 4. The dependence of the error on the value of $k\rho_0$

To determine the accuracy of the representation of fields (1) by this multipole, the dependence of the average relative error [6] of expressions (18) and (19) on the value $k\rho_0$, has been studied. These expressions have the same errors, therefore, for definiteness, only expression (18) is considered. Graphs of the resulting error for the initial values of n, are shown on Figure 4. Despite the fact that the error for n = 1 is greater than for n = 2 and n = 3, it gradually increases with increasing n. In addition, the size of the multipole also increases.

Thus, fields (1) can be approximately described by the first type circular multipole. At the same time, to ensure the same accuracy as n increases, the value of ρ_0 (determining the size of the multipole) should be reduced.

4. SECOND TYPE CIRCULAR MULTIPOLE

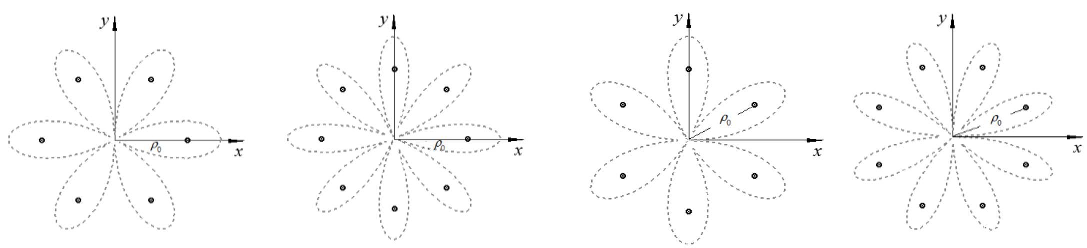
The other multipole type can be obtained from the expressions (16) and (17), if we notice that the inner sums of their left parts represent the first n+1 terms of the Taylor series of functions $H_0^{(1)}\left(k\left|\vec{\rho}-\rho_0\vec{\rho}_{n,j}\right|\right)$ and $H_0^{(1)}\left(k\left|\vec{\rho}-\rho_0\vec{\rho}^*_{n,j}\right|\right)$. After appropriate substitution, for the area $\rho > \rho_0$, approximately we will have

$$\frac{2^{n-2}(n-1)!}{(k\rho_0)^n}\sum_{j=1}^{2n}(-1)^{j+1}H_0^{(1)}\left(k\left|\vec{\rho}-\rho_0\vec{\rho}_{n,j}\right|\right)\approx H_n^{(1)}(k\rho)\cos(n\varphi)\,, \tag{20}$$

$$\frac{2^{n-2}(n-1)!}{(k\rho_0)^n}\sum_{j=1}^{2n}(-1)^{j+1}H_0^{(1)}\left(k\left|\vec{\rho}-\rho_0\vec{\rho}^{\,*}_{n,j}\right|\right)\approx H_n^{(1)}(k\rho)\sin(n\varphi). \quad (21)$$

Analyzing these approximate equalities, we come to the conclusion that $2n$ monopoles, with radius vectors $\rho_0\vec{\rho}_{n,j}$ and $\rho_0\vec{\rho}^{\,*}_{n,j}$, radiate the total field of the form $H_n^{(1)}(k\rho)\cos(n\varphi)$ and $H_n^{(1)}(k\rho)\sin(n\varphi)$ respectively. These monopoles are located on a circle with a radius ρ_0 and their amplitudes are $2^{n-2}(n-1)!(k\rho_0)^{-n}(-1)^{j+1}$. The multiplier $(-1)^{j+1}$ shows that each subsequent monopole oscillates in the opposite phase to the previous one.

The structure of the multipole that emerges, referred to as the second type circular multipole, is illustrated in Figure 5. It can be seen that the number of monopoles coincides with the number of original field pattern lobes.



(a) (b)

Figure 5. The second type of circular multipole, which describes the field: (a) $H_n^{(1)}(k\rho)\cos(n\varphi)$ and (b) $H_n^{(1)}(k\rho)\sin(n\varphi)$, when n=3and n=4

Figure 6 presents the dependence of the average relative error of expression (20) on the value $k\rho_0$, for several initial values of n. It can be seen that the greatest error is observed in case when n = 1, and for subsequent values of n, unlike the previous case (Figure 4), the error, on the contrary, decreases. So, starting from n = 3, in the given scales of the figure, the corresponding error graphs actually essentially merge with the horizontal axis. This indicates that the second type multipole represents the original fields with greater accuracy. Moreover, the number of monopoles reduces, making it more efficient than the first type multipole.

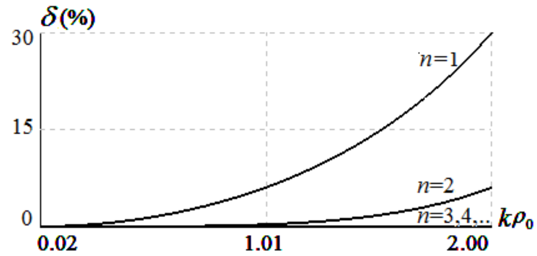


Figure 6. The dependence of the error on the value of $k\rho_0$

5. APPLICATION OF THE ADDITION THEOREM

Let us present the mathematical expression of the well-known addition theorem [1, 8] for cylindrical functions,

$$H_0^{(1)}\left(k\left|\vec{\rho}-\vec{\rho}_0\right|\right)=J_0\left(k\rho_0\right)H_0^{(1)}\left(k\rho\right)+2\sum_{m=1}^{\infty}J_m\left(k\rho_0\right)H_m^{(1)}\left(k\rho\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]. \qquad (22)$$

Here $\vec{\rho}_0$ is a radius vector of point with coordinates (ρ_0, φ_0), located on a circle l_0, radius ρ_0 and centered at the origin. It is assumed that the observation point (ρ, φ) is located in outer area ($\rho > \rho_0$). In outer area the considered row converges uniformly. Next, two more types of multipoles, that describe the fields (1) are obtained, based on the application of expression (22).

5.1. Third Type of Circular Multipole

Let's multiply both sides of (22) by $\cos\left(n\varphi_0\right)$ ($\sin\left(n\varphi_0\right)$) and integrate along the circle l_0. After a series of calculations, we obtain the expressions

$$\left[2\pi\rho_0 J_n\left(k\rho_0\right)\right]^{-1}\int_{l_0}H_0^{(1)}\left(k\left|\vec{\rho}-\vec{\rho}_0\right|\right)\cos\left(n\varphi_0\right)dl_0=H_n^{(1)}\left(k\rho\right)\cos\left(n\varphi\right), \qquad (23)$$

$$\left[2\pi\rho_0 J_n\left(k\rho_0\right)\right]^{-1}\int_{l_0}H_0^{(1)}\left(k\left|\vec{\rho}-\vec{\rho}_0\right|\right)\sin\left(n\varphi_0\right)dl_0=H_n^{(1)}\left(k\rho\right)\sin\left(n\varphi\right). \qquad (24)$$

Let us replace the integrals in (23) and (24) with the corresponding integral sums. If N is a sufficiently large number, then as a result we obtain approximate equalities

$$\left[NJ_n\left(k\rho_0\right)\right]^{-1}\sum_{j=1}^{N}H_0^{(1)}\left(k\left|\vec{\rho}-\rho_0\vec{\rho}_{N,j}\right|\right)\cos\left(n\varphi_{N,j}\right)\approx H_n^{(1)}\left(k\rho\right)\cos\left(n\varphi\right), \qquad (25)$$

$$\left[NJ_n\left(k\rho_0\right)\right]^{-1}\sum_{j=1}^{N}H_0^{(1)}\left(k\left|\vec{\rho}-\rho_0\vec{\rho}^{*}_{N,j}\right|\right)\sin\left(n\varphi^{*}_{N,j}\right)\approx H_n^{(1)}\left(k\rho\right)\sin\left(n\varphi\right), \qquad (26)$$

where

$$\vec{\rho}_{N,j} = \left\{\cos\varphi_{N,j}, \sin\varphi_{N,j}\right\},\ \vec{\rho}^{\,*}_{N,j} = \left\{\cos\varphi^*_{N,j}, \sin\varphi^*_{N,j}\right\},$$

$$\varphi_{N,j} = \frac{2\pi}{N}(j-1),\ \varphi^*_{N,j} = \frac{2\pi}{N}\left(j-\frac{1}{2}\right),\ j=1,...,N\ .$$

Consequently, a set of N monopoles on a circle of radius ρ_0, having amplitudes $\left[NJ_n(k\rho_0)\right]^{-1}$, emit a total field of type (1) in the outer region ($\rho > \rho_0$). The difference between the resulting third type of multipole and the second is the number N of its monopoles, which in this case is not limited. It's interesting to compare their accuracy.

For this purpose, let us denote the left and right parts of expressions (25) and (26), respectively, as $L^{III}_{N,n}(\vec{\rho})$, $R_n(\vec{\rho})$ and $L^{III*}_{N,n}(\vec{\rho})$, $R^*_n(\vec{\rho})$. Then we can briefly write

$$L^{III}_{N,n}(\vec{\rho}) \approx R_n(\vec{\rho}),\ L^{III*}_{N,n}(\vec{\rho}) \approx R^*_n(\vec{\rho}).$$

Let's consider a special case when N= 2n. Note that then

$$\varphi_{N,j} = \varphi_{n,j},\ \varphi^*_{N,j} = \varphi^*_{n,j},\ \vec{\rho}_{N,j} = \vec{\rho}_{n,j},\ \vec{\rho}^{\,*}_{N,j} = \vec{\rho}^{\,*}_{n,j},\ \cos\left(n\varphi_{N,j}\right) = \sin\left(n\varphi^*_{N,j}\right) = (-1)^{j+1}.$$

In addition, if we assume that the value of $k\rho_0$ (the perimeter of the multipole in units of wavelength) is so small that inequality $0 < k\rho_0 \le \sqrt{n+1}$ holds, then we can use the well-known asymptotic expression [7]

$$J_n(k\rho_0) \approx \frac{1}{n!}\left(\frac{k\rho_0}{2}\right)^n.$$

As a result, the left sides $L^{III}_{N,n}(\vec{\rho})$ and $L^{III*}_{N,n}(\vec{\rho})$, for N= 2n, will be transformed to the form

$$L^{III}_{N,n}(\vec{\rho})\Big|_{N=2n} = \frac{2^{n-1}(n-1)!}{(k\rho_0)^n}\sum_{j=1}^{2n}(-1)^{j+1} H^{(1)}_0\left(k\left|\vec{\rho}-\rho_0\vec{\rho}_{n,j}\right|\right),$$

$$L^{III*}_{N,n}(\vec{\rho})\Big|_{N=2n} = \frac{2^{n-1}(n-1)!}{(k\rho_0)^n}\sum_{j=1}^{N}(-1)^{j+1} H^{(1)}_0\left(k\left|\vec{\rho}-\rho_0\vec{\rho}^{\,*}_{n,j}\right|\right).$$

Comparing them with the left sides $L^{II}_n(\vec{\rho})$ and $L^{II*}_n(\vec{\rho})$ of expressions (20) and (21) we notice that

$$L^{III}_{N,n}(\vec{\rho})\Big|_{N=2n} = 2L^{II}_n(\vec{\rho}),\ L^{III*}_{N,n}(\vec{\rho})\Big|_{N=2n} = 2L^{II*}_n(\vec{\rho}).$$

But on the other hand, according to (20) and (21), we have

$$L^{II}_n(\vec{\rho}) \approx R_n(\vec{\rho}),\ L^{II*}_n(\vec{\rho}) \approx R^*_n(\vec{\rho}),$$

from which

$$L_{N,n}^{III}(\vec{\rho})\Big|_{N=2n} \approx 2R_n(\vec{\rho}),\ L_{N,n}^{III*}(\vec{\rho})\Big|_{N=2n} \approx 2R_n^*(\vec{\rho}).$$

Thus, for $N = 2n$, approximate equalities (25) and (26) are violated. For the relative error at the considered point of the plane, we will have

$$\delta(\vec{\rho}) = \frac{L_{N,n}^{III}(\vec{\rho})\Big|_{N=2n} - R_n(\vec{\rho})}{R_n(\vec{\rho})}100\% \approx \frac{2R_n(\vec{\rho}) - R_n(\vec{\rho})}{R_n(\vec{\rho})}100\% = 100\%,$$

$$\delta^*(\vec{\rho}) = \frac{L_{N,n}^{III*}(\vec{\rho})\Big|_{N=2n} - R_n^*(\vec{\rho})}{R_n^*(\vec{\rho})}100\% \approx \frac{2R_n^*(\vec{\rho}) - R_n^*(\vec{\rho})}{R_n^*(\vec{\rho})}100\% = 100\%.$$

Consequently, to ensure sufficient accuracy of the representation of fields (1) by this type of circular multipole, the number N of its monopoles must satisfy condition $N > 2n$. This makes it non-optimal, in comparison with the circular multipole of the second type.

5.2. The Linear Multipole

Let us consider the addition theorem (22) again. Through numerical calculations one can be convinced that if the value of $k\rho_0$ satisfies the condition

$$k\rho_0 \le n/4, \qquad (27)$$

then with sufficient accuracy we can limit ourselves to the first n terms of the series. Let us assume that$\varphi_0 = 0$ and condition (27) is satisfied. Then we can write

$$H_0^{(1)}(k|\vec{\rho}-\vec{\rho}_0|) \approx J_0(k\rho_0)H_0^{(1)}(k\rho) + 2\sum_{m=1}^{n} J_m(k\rho_0)H_m^{(1)}(k\rho)\cos(m\varphi),\ (\rho > \rho_0). \qquad (28)$$

Let us now consider $n+1$ points on the abscissa axis, with radius vectors

$$\vec{\rho}_{n,\alpha} = \{(\alpha - n/2)d_n, 0, 0\},\ \alpha = 0,...,n \qquad (29)$$

Notice, that if n is odd number, then for the given points, the condition $|\vec{\rho}_{n,\alpha}| \ne 0$, $\alpha = 0,...,n$ will take a place. If n is even number, then $|\vec{\rho}_{n,n/2}| = 0$. We choose the distance d_n between neighboring points in such a way that $kd_n = 1/2$. In this case, for all values of the vectors $\vec{\rho}_{n,\alpha}$, the inequality $k|\vec{\rho}_{n,\alpha}| \le n/4$ will take place. Let's say n is an odd number. Then, by virtue of (27) and (28), we write

$$H_0^{(1)}(k|\vec{\rho}-\vec{\rho}_{n,\alpha}|) \approx J_0(k|\vec{\rho}_{n,\alpha}|)H_0^{(1)}(k\rho) + 2\sum_{m=1}^{n} J_m(k|\vec{\rho}_{n,\alpha}|)H_m^{(1)}(k\rho)\cos(m\varphi_{n,\alpha}),$$

where $\varphi_{n,\alpha}$ is an angle between $\vec{\rho}_{n,\alpha}$ and $\vec{\rho}$ vectors. If n is an even number, then the angle $\varphi_{n,\alpha}$ loses its meaning at $\alpha = n/2$. Therefore, in this case we write

$$H_0^{(1)}\left(k\left|\vec{\rho}-\vec{\rho}_{n,\alpha}\right|\right)\Big|_{\alpha\neq n/2} \approx J_0\left(k\left|\vec{\rho}_{n,\alpha}\right|\right)\Big|_{\alpha\neq n/2} H_0^{(1)}(k\rho) + 2\sum_{m=1}^{n} J_m\left(k\left|\vec{\rho}_{n,\alpha}\right|\right)\cos\left(m\varphi_{n,\alpha}\right)\Big|_{\alpha\neq n/2} H_m^{(1)}(k\rho),$$

$$H_0^{(1)}\left(k\left|\vec{\rho}-\vec{\rho}_{n,\alpha}\right|\right)\Big|_{\alpha=n/2} = H_0^{(1)}(k\rho).$$

Let's introduce unknown quantities $A_{n,\alpha}$ and compose a sum of the form

$$\sum_{\alpha=0}^{n}(-1)^{\alpha} A_{n,\alpha} H_0^{(1)}\left(k\left|\vec{\rho}-\vec{\rho}_{n,\alpha}\right|\right). \tag{30}$$

Based on the previous expressions, we can write

$$\sum_{\alpha=0}^{n}(-1)^{\alpha} A_{n,\alpha} H_0^{(1)}\left(k\left|\vec{\rho}-\vec{\rho}_{n,\alpha}\right|\right) \approx S_{0,n} H_0^{(1)}(k\rho) + 2\sum_{m=1}^{n} S_{m,n} H_m^{(1)}(k\rho)\cos(m\varphi). \tag{31}$$

The quantities $S_{0,n}$ and $S_{m,n}$, for odd n, are determined as

$$S_{0,n} = \sum_{\alpha=0}^{n}(-1)^{\alpha} A_{n,\alpha} J_0\left(k\left|\vec{\rho}_{n,\alpha}\right|\right),\ S_{m,n} = \sum_{\alpha=0}^{n}(-1)^{\alpha} A_{n,\alpha} J_m\left(k\left|\vec{\rho}_{n,\alpha}\right|\right)\cos\left(m\varphi_{n,\alpha}\right).$$

For even n, we respectively have

$$S_{0,n} = (-1)^{n/2} A_{n,n/2} + \sum_{\substack{\alpha=0 \\ \alpha\neq n/2}}^{n}(-1)^{\alpha} A_{n,\alpha} J_0\left(k\left|\vec{\rho}_{n,\alpha}\right|\right),\ S_{m,n} = \sum_{\substack{\alpha=0 \\ \alpha\neq n/2}}^{n}(-1)^{\alpha} A_{n,\alpha} J_m\left(k\left|\vec{\rho}_{n,\alpha}\right|\right)\cos\left(m\varphi_{n,\alpha}\right).$$

Taking into account the obvious equalities

$$\left|\vec{\rho}_{n,\alpha}\right| = \left|\vec{\rho}_{n,n-\alpha}\right|,\ \cos\left(m\varphi_{n,n-\alpha}\right) = \cos(m\varphi),\ \cos\left(m\varphi_{n,\alpha}\right) = \cos\left[m(\pi-\varphi)\right] = (-1)^m\cos(m\varphi),$$

expressions for $S_{0,n}$ and $S_{m,n}$ can be written in a more convenient form. So, for odd n,

$$S_{0,n} = \sum_{\alpha=0}^{(n-1)/2}(-1)^{\alpha}\left(A_{n,\alpha} - A_{n,n-\alpha}\right) J_0\left(k\left|\vec{\rho}_{n,\alpha}\right|\right),$$

$$S_{m,n} = \sum_{\alpha=0}^{(n-1)/2}(-1)^{\alpha}\left[(-1)^m A_{n,\alpha} - A_{n,n-\alpha}\right] J_m\left(k\left|\vec{\rho}_{n,\alpha}\right|\right)$$

and for even n, respectively

$$S_{0,n} = (-1)^{n/2} A_{n,n/2} + \sum_{\alpha=0}^{(n/2)-1}(-1)^{\alpha}\left(A_{n,\alpha} + A_{n,n-\alpha}\right) J_0\left(k\left|\vec{\rho}_{n,\alpha}\right|\right),$$

$$S_{m,n} = \sum_{\alpha=0}^{(n/2)-1}(-1)^{\alpha}\left[(-1)^m A_{n,\alpha} + A_{n,n-\alpha}\right] J_m\left(k\left|\vec{\rho}_{n,\alpha}\right|\right).$$

We now require that the quantities $S_{0,n}$ and $S_{m,n}$ satisfy the conditions

$$S_{0,n} = 0 \, , \; S_{m,n} = \frac{1}{2}\delta_{mn} \, . \tag{32}$$

In this case, from expression (31), taking into account (29), we obviously obtain

$$\sum_{\alpha=0}^{n} (-1)^{\alpha} A_{n,\alpha} H_0^{(1)}\left(k\left| \vec{\rho} - \left(\alpha - \frac{n}{2}\right) d_n \vec{x} \right| \right) \approx H_n^{(1)}(k\rho)\cos(n\varphi) \, . \tag{33}$$

Note that if we consider $n+1$ points on the ordinate axis, with radius vectors

$$\vec{\rho}^{\,*}_{n,\alpha} = \{0, (\alpha - n/2) d_n, 0\} \, , \; \alpha = 0, \ldots, n \, , \tag{34}$$

then, similarly to (33), we obtain

$$\sum_{\alpha=0}^{n} (-1)^{\alpha} A_{n,\alpha} H_0^{(1)}\left(k\left| \vec{\rho} - \left(\alpha - \frac{n}{2}\right) d_n \vec{y} \right| \right) \approx H_n^{(1)}(k\rho)\sin(n\varphi) \, . \tag{35}$$

Thus, the problem is reduced to determining the coefficients $A_{n,\alpha}$, for which we use conditions (32). These conditions, in expanded form, taking into account the expressions for $S_{0,n}$ and $S_{m,n}$, will be written for odd *n* as

$$\sum_{\alpha=0}^{(n-1)/2} (-1)^{\alpha} \left[(-1)^m A_{n,\alpha} - A_{n,n-\alpha} \right] J_m\left(k\left| \vec{\rho}_{n,\alpha} \right| \right) = \frac{1}{2}\delta_{mn} \, , \; m = 0, \ldots, n \, ,$$

and for even *n*, respectively as

$$\begin{cases} (-1)^{n/2} A_{n,n/2} + \sum\limits_{\alpha=0}^{(n/2)-1} (-1)^{\alpha} \left(A_{n,\alpha} + A_{n,n-\alpha} \right) J_0\left(k\left| \vec{\rho}_{n,\alpha} \right| \right) = 0 \\ \sum\limits_{\alpha=0}^{(n/2)-1} (-1)^{\alpha} \left[(-1)^m A_{n,\alpha} + A_{n,n-\alpha} \right] J_m\left(k\left| \vec{\rho}_{n,\alpha} \right| \right) = \frac{1}{2}\delta_{m,n} \end{cases} , \; m = 1, \ldots, n \, .$$

These expressions represent a system of linear algebraic equations with respect to unknown coefficients $A_{n,\alpha}$. The number of equations, as well as unknowns, is $n+1$. Note that through a series of transformations of these systems, this number can be reduced. Thus, for odd *n*, the first $(n+1)/2$ unknowns can be found by solving the system

$$\sum_{\alpha=0}^{(n-1)/2} (-1)^{\alpha} J_{2\beta+1}\left(k\left| \vec{\rho}_{n,\alpha} \right| \right) A_{n,\alpha} = -\frac{1}{4}\delta_{\beta,(n-1)/2} \, , \; \beta = 0, \ldots, (n-1)/2 \tag{36}$$

and the remaining ones, express through them, through equalities

$$A_{n,n-\alpha} = A_{n,\alpha} \, , \; \beta = 0, \ldots, (n-1)/2 \, . \tag{37}$$

Similarly, for even *n*, the first $n/2$ unknowns can be determined from the system

$$\sum_{\alpha=0}^{(n/2)-1}(-1)^{\alpha}J_{2\beta}\left(k\left|\vec{\rho}_{n,\alpha}\right|\right)A_{n,\alpha}=\frac{1}{4}\delta_{\beta,n/2},\ \beta=1,...,n/2, \tag{38}$$

then determine the coefficient $A_{n,n/2}$ as

$$A_{n,n/2}=2\sum_{\alpha=0}^{(n/2)-1}(-1)^{\alpha+(n/2)-1}J_0\left(k\left|\vec{\rho}_{n,\alpha}\right|\right)A_{n,\alpha} \tag{39}$$

and the remaining $n/2$ unknowns, determine from the equalities

$$A_{n,n-\alpha}=A_{n,\alpha},\ \alpha=0,...,n/2-1. \tag{40}$$

The left parts of expressions (33) and (35) describe multipoles consisting of $n+1$ monopoles located, respectively, along the abscissa and ordinate axis, at points with radius vectors (29) and (34). Moreover, if the amplitudes $A_{n,\alpha}$ of these monopoles satisfy conditions (36), (37) or (38)-(40), depending on the parity of n, then their total fields coincide with the original fields (1) with sufficient accuracy. This type of multipole can be called linear.

The length of the considered multipole, for a given n, is determined as $l_n = nd_n$. Above we assumed that $kd_n=1/2$. If we reduce the value of kd_n (and therefore the length of the multipole), then the accuracy of expressions (33) and (35) will increase. It is interesting to compare the accuracy of the representation of fields (1) by a linear multipole and a circular multipole of the second type. Let us determine the radius ρ_0 of the circular multipole as $\rho_0=n/(4k)$. Then its diameter will be equal to the length of the linear multipole (Figure 7).

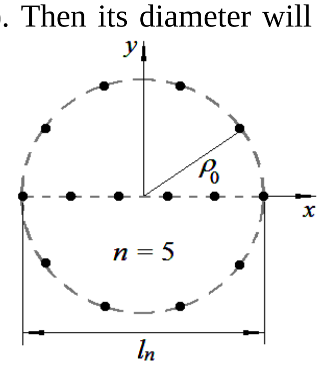


Figure 7. The second type of circular multipole and linear multipole

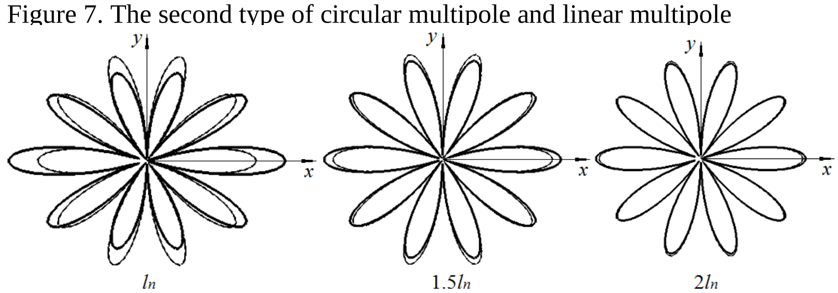


Figure 8. Comparison of amplitude radiation patterns of linear and circular multipoles

Figure 8 presents a comparison of the radiation patterns of the fields from the two types of multipoles under consideration, with the corresponding patterns of the field $H_n^{(1)}(k\rho)\cos(n\varphi)$, for n = 5. These radiation patterns are constructed at three different distances from the center of the multipoles (l_n, $1.5l_n$ and $2l_n$). The patterns of linear and circular multipoles are marked with a thick and thin continuous line, respectively. The pattern of the field $H_n^{(1)}(k\rho)\cos(n\varphi)$ is marked with a dotted line. The thin continuous line is so close to the dashed line that at the given scales, they are indistinguishable. This demonstrates the high accuracy with which the second type circular multipoles describe fields (1). In contrast, the linear multipole exhibits comparatively lower accuracy, which, however, improves with increasing distance.

6. CONCLUSIONS

The structures of field sources described by wave functions (1) of a circular cylinder have been examined. By analyzing the radiation patterns of field (1) and applying the addition theorem for cylindrical functions, four types of multipoles have been identified. These vary in their structure, the number of monopoles, and the precision with which they represent the original fields. From the findings, it is concluded that a second type circular multipole, consisting of 2n monopoles, achieves higher accuracy. However, a smaller linear multipole, comprising n+1 monopoles, may offer greater optimality.

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